OLD AND NEW GENERALIZATIONS OF CLASSICAL TRIANGLE CENTRES TO TETRAHEDRA

TEAM MEMBERS
Trevor Kai Hei CHEUNG, Hon Ching KO

TEACHER
Mr. Pak Leong CHEUNG

SCHOOL
St. Paul’s Co-educational College

Abstract. The classical triangle centres, namely centroid, circumcentre, incentre, excentre, orthocentre and Monge point, will be generalized to tetrahedra in a unified approach as points of concurrence of special lines. Our line characterization approach will also enable us to create new tetrahedron centres lying on the Euler lines, which will be a family with nice geometry including Monge point and twelve-point centre.

Two tetrahedron centres generalizing orthocentre of triangles from new perspectives will be constructed through introducing antimedial tetrahedra, tangential tetrahedra and a new kind of orthic tetrahedra. The first one, defined as the circumcentre of the antimedial tetrahedron of a tetrahedron, will be proved to lie on the Euler line. The second one, defined as the incentre or a suitable excentre of the new orthic tetrahedron of a tetrahedron, will be discovered to be collinear with its circumcentre and twenty-fifth Kimberling centre $\chi_{25}$. Surprisingly, these two differently motivated geometric generalizations turn out to have analogous algebraic representations.

A clear definition of tetrahedron centres, as a generalization of triangle centres to tetrahedra, will be coined to set up a framework for studying analogies between geometries of triangles and tetrahedra. Fundamental properties of tetrahedron centres will be studied.

1. Introduction

1.1. Generalizing Triangle Centres to Tetrahedra (and Higher-Dimensional Simplices)

The classical triangle centres, namely centroid, circumcentre, incentre, excentre and orthocentre, have been a special part of high school geometry syllabi. The unique feature that they are respectively the points of concurrence of medians, perpendicular bisectors, interior angle bisectors, exterior angle bisectors and altitudes of
triangles has fascinated many mathematics lovers. See [14] and [15] for a framework of triangle centres.

Tetrahedra have been the most straightforward generalization of triangles to the three-dimensional space. It is therefore reasonable to investigate how much geometry of triangles carries over or does not carry over to them. Generalizing triangle centres to tetrahedra has been one fruitful aspect of this general theme. See [11] for an excellent survey.

It is imaginable that the first four classical triangle centres can be naturally and satisfactorily generalized to tetrahedra as tetrahedron centres. However, since the altitudes of a tetrahedron may not be concurrent, an orthocentre may not exist; if it exists, the tetrahedron is said to be orthocentric. But then an innovative generalization called Monge point came to rescue such oddity: it exists uniquely in any tetrahedron, and coincides with the orthocentre in any orthocentric tetrahedron. See [3], [6], [11], [13], [16], [17] and [18].

Some non-classical triangle centres have also been satisfactorily generalized to tetrahedra. They include nine-point centre (generalized to twelve-point centre), symmedian point (a.k.a. Lemoine point and Grebe point), Gergonne point, Nagel point, Spieker centre (a.k.a. cleavage centre) and Fermat-Torricelli point (a.k.a. first isogonic centre when no angle exceeds $2\pi/3$). Indeed, all the above tetrahedron centres can be further generalized to higher-dimensional simplices. See [1], [2], [3], [4], [9], [12], [18], [19] and [20]. Nevertheless, the generalized triangle centres are only front members $\chi_n$, $n = 1, 2, 3, 4, 5, 6, 7, 8, 10, 13$, of Kimberling centres [15], and still a lot more have not been generalized yet.

1.2. Aims, Objectives and Organization

This paper will be devoted to

(a) showing line characterizations of the classical tetrahedron centres,
(b) generalizing Monge point of tetrahedra to a family of tetrahedron centres lying on the Euler line,
(c) generalizing orthocentre of triangles to tetrahedra in two new ways, and
(d) formulating a framework of tetrahedron centres.

We will take a coordinate-free analytic approach which requires only basic linear algebra, whereas high school level synthetic proofs will be provided as well for some of the results.

Traditionally, the classical tetrahedron centres are defined or characterized as points of concurrence of special planes of tetrahedra (cf. [3], [6], [13], [16], [17], [18]). In Section 2, we will characterize them as points of concurrence of special lines of tetrahedra instead. The theory will become better in three aspects because of this
line characterization approach: (i) lines are simpler geometric objects than planes, making ideas easier to visualize, (ii) there are fewer special lines than special planes, making pictures less complicated and messy, and (iii) the uniqueness of a common point of lines is usually more obvious than that of planes.

These line characterizations are also main properties of the classical triangle centres that carry over to tetrahedra during the generalizations, and will also motivate and supply useful tools for the latter sections. Although [11] commented that ‘The definitions of these three centers [centroid, circumcentre and incentre] and most of their main properties can be carried over to tetrahedra [...] in a very natural manner, and proofs are often routine generalizations.’, we found that a rigorous and meticulous treatment actually requires quite a lot of effort and care. Excentres has been defined in [21], but our description may look clearer.

Among the classical tetrahedron centres, Monge point is a special one. However, in Section 3, we will discover that it is just a member of a vast family of tetrahedron centres lying on the Euler line. This family of tetrahedron centres will be named as quasi-orthocentres, since they will be characterized as the points of concurrence of special lines sharing some common properties with altitudes.

While Monge point is the most recognized generalization of orthocentre of triangles, we will present two more generalizations in Section 4, as orthocentre of triangles possesses a range of characterizations. They will be named as antimedial circumcentre and orthic inexcentre, as they will be constructed through antimedial tetrahedra, tangential tetrahedra and a new kind of orthic tetrahedra. We will discover homothety between a tetrahedron and its antimedial tetrahedron, and between the tangential tetrahedron and the orthic tetrahedron of the tetrahedron. We will then derive geometric and algebraic properties of antimedial circumcentre and orthic inexcentre that are properties of orthocentre of triangles that carry over to tetrahedra during these two generalizations. These results will be closely related to the homotheties. The twenty-fifth Kimberling centre $\chi_{25}$ of triangles will also be generalized as a by-product.

Despite we have kept talking about tetrahedron centre, it seems that this terminology has been used without a clear definition. Through having generalized various triangle centres to tetrahedra in the preceding sections, we will have come up with the elements needed for defining this general terminology. In Section 5, we will suggest our definition, and show that all the tetrahedron centres in this paper fulfill the requirements. Finally, we will prove a couple of simple ways to construct tetrahedron centres from others, and will show how all tetrahedron centres can be expressed in terms of barycentric coordinates.
1.3. Terminologies and Notations

A tetrahedron \( \Delta(V)/\Delta(V_0, V_1, V_2, V_3)/[V_0, V_1, V_2, V_3] \) in the three-dimensional Euclidean space \( \mathbb{R}^3 \) is the convex hull

\[
[v_0, v_1, v_2, v_3] := \{\lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 : \lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = 1
\text{ and } 0 \leq \lambda_0, \lambda_1, \lambda_2, \lambda_3 \leq 1\} \quad (1)
\]

of its vertex set \( V = \{V_0, V_1, V_2, V_3\} \), where \( v_0, v_1, v_2, v_3 \) are the position vectors of the vertices \( V_0, V_1, V_2, V_3 \). The edge joining the vertices \( V_i \) and \( V_j \) is the convex hull \( E_{i,j} := [V_i, V_j] \) of them, whose direction vector is \( e_{i,j} := v_i - v_j \). The face opposite to the vertex \( V_i \) is the convex hull \( F_i := [V \setminus \{V_i\}] \) of all the vertices except \( V_i \).

The vertices \( V_0, V_1, V_2, V_3 \) have to be non-coplanar in order to form a tetrahedron, which is equivalent to requiring that \( \{v_0, v_1, v_2, v_3\} \) are affinely independent, i.e. \( \{v_1 - v_0, v_2 - v_0, v_3 - v_0\} \) are linearly independent.

With this affine independence, the edges \( E_{i,j} \) and the faces \( F_i \) are guaranteed to be line segments and triangles, and the position vector of every point in the tetrahedron can be expressed uniquely as a convex combination of the form (1).

Throughout this paper, points (as geometric objects) and their position vectors (as algebraic objects on which we can perform operations) will be used interchangeably. For ease of readability, we will adopt the notational convention that while points will be denoted by italic uppercase letters \( A, B, C, \ldots \) or \( \Delta, \mathcal{B}, \mathcal{C}, \ldots \), their position vectors will be denoted by the corresponding boldface lowercase letters \( a, b, c, \ldots \), and the special lines associated to these points will be denoted by \( a, b, c, \ldots \).

2. Classical Tetrahedron Centres

First of all, let us recall the classical triangle centres and their main properties through the following figures:
In this section, we will use a unified approach — generalize the vertices of a triangle as the vertices of a tetrahedron, and generalize the edges of a triangle as the faces of a tetrahedron — to generalize the classical triangle centres to tetrahedra. We will define centroid, circumcentre, incentre, excentre, orthocentre and Monge point of tetrahedra, and characterize them as the points of concurrence of special lines of tetrahedra. We will also show that the geometric properties of their two-dimensional counterparts as shown in Figures 1, 2, 3, 4 and 5 are retained when generalized to tetrahedra.
2.1. Centroid

**Definition 1. (Centroid and median)** Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. The centroid $\mathcal{G}$ of $\Delta$ is defined as

$$g = \frac{1}{4}(v_0 + v_1 + v_2 + v_3) \quad (2)$$

A median $g_i$ of $\Delta$ is the line passing through $V_i$ and $G_i$, where $G_i$ abbreviates the centroid $\mathcal{G}(F_i)$ of the face $F_i$.

We now re-prove (cf. Commandino’s theorem) that centroid can be characterized as the point of concurrence of medians. See Figure 6 for an illustration.

**Proposition 2.** (Centroid as point of concurrence of medians) The centroid $\mathcal{G}$ of a tetrahedron $\Delta = [V_0, V_1, V_2, V_3]$ is the point of concurrence of its four medians. It divides each median segment $[V_i, G_i]$ internally in $V_iG : GG_i = 3 : 1$.

**Proof.** Let $\{i, j, k, l\} = \{0, 1, 2, 3\}$. Then,

$$g = \frac{1}{4}(v_0 + v_1 + v_2 + v_3) = \frac{1}{4}v_i + \frac{1}{4}(v_j + v_k + v_l) = \frac{1}{4}v_i + \frac{3}{4}g_i,$$

showing that $\mathcal{G}$ lies on $g_i$ and divides the median segment $[V_i, G_i]$ internally in the claimed ratio.

Moreover, the medians cannot intersect at more points, otherwise, they cannot be distinct lines since two points determine a line.
2.2. Circumcentre

Definition 3. (Circumcentre and perpendicular bisector) Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. The circumcentre $O$ of $\Delta$ is the unique point $o$ equidistant to its vertices, i.e.

$$||o - v_0|| = \cdots = ||o - v_3||$$

The common distance in (3) is called the circumradius of $\Delta$, denoted by $R(\Delta)$. The sphere

$$S^{ci}(\Delta) := \{x : ||x - o|| = R(\Delta)\}$$

is called the circumsphere of $\Delta$.

A perpendicular bisector $o_i$ of $\Delta$ is the line passing through $O_i$ and perpendicular to $F_i$, where $O_i$ abbreviates the circumcentre $O(F_i)$ of the face $F_i$.

We now prove the existence and uniqueness of the solution of (3) in order that circumcentre, circumradius and circumsphere are well-defined, as well as that circumcentre can be characterized as the point of concurrence of perpendicular bisectors. See Figure 7 for an illustration.

Proposition 4. (Circumcentre as point of concurrence of perpendicular bisectors) Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. Then, equation 3 has a unique solution $o$.

Moreover, the circumcentre $O$ of $\Delta$ is the point of concurrence of its four perpendicular bisectors. Consequently, the projection of $O$ onto a face $F_i$ is precisely the circumcentre $O_i$ of the face.
Proof. The equations captured by (3) are

\[ \| o - v_0 \|^2 = \| o - v_i \|^2 \]
\[ \| o \|^2 - 2 o \cdot v_0 + \| v_0 \|^2 = \| o \|^2 - 2 o \cdot v_i + \| v_i \|^2 \]
\[ (v_i - v_0) \cdot o = \frac{1}{2}(\| v_i \|^2 - \| v_0 \|^2) \quad \text{for } i = 1, 2, 3. \] (4)

Writing \( o = (x_1, x_2, x_3) \) and \( v_i - v_0 = (a_{i,1}, a_{i,2}, a_{i,3}) \), (4) becomes

\[ a_{i,1}x_1 + a_{i,2}x_2 + a_{i,3}x_3 = b_i, \]

where \( b_i = \frac{1}{2}(\| v_i \|^2 - \| v_0 \|^2) \), and we obtain the system of linear equations

\[
\begin{pmatrix}
  a_{1,1} & a_{1,2} & a_{1,3} \\
  a_{2,1} & a_{2,2} & a_{2,3} \\
  a_{3,1} & a_{3,2} & a_{3,3}
\end{pmatrix}
\begin{pmatrix}
  x_1 \\
  x_2 \\
  x_3
\end{pmatrix}
=
\begin{pmatrix}
  b_1 \\
  b_2 \\
  b_3
\end{pmatrix}
\]

Since \( \{v_1 - v_0, v_2 - v_0, v_3 - v_0\} \) are linearly independent, the coefficient matrix will be invertible, and the system has a unique solution, thus proving the required existence and uniqueness.

Let \( \{i, j, k, l\} = \{0, 1, 2, 3\} \), and consider the circumcentre \( O_i/o_i \) of the face \( F_i \), which is the point in the plane containing \( F_i \) that satisfies

\[ \| o_i - v_j \| = R_i > 0 \]
\[ \| o_i \|^2 - 2 o_i \cdot v_j + \| v_j \|^2 = R_i^2 \] (5)

To show the required concurrence of the perpendicular bisectors, it suffices to show that

\[ (o - o_i) \cdot (v_j - v_k) = 0 \] (6)
so that the line joining $O$ and $O_i$ is perpendicular to $F_i$ and so $O$ lies on $o_i$. To this end, use (4) to get
\[
0 \cdot (v_j - v_k) = 0 \cdot (v_j - v_0) - 0 \cdot (v_k - v_0)
= \frac{1}{2} (||v_j||^2 - ||v_0||^2) - \frac{1}{2} (||v_k||^2 - ||v_0||^2)
= \frac{1}{2} (||v_j||^2 - ||v_k||^2)
\]
and use (5) to get
\[
o_i \cdot (v_j - v_k) = \frac{1}{2} (||o_i||^2 + ||v_j||^2 - R_i^2) - \frac{1}{2} (||o_i||^2 + ||v_k||^2 - R_i^2)
= \frac{1}{2} (||v_j||^2 - ||v_k||^2)
\]
Then, (6) follows from subtracting (8) from (7).

2.3. Incentre and Excentre

**Definition 5. (Incentre and interior angle bisector)** Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. For $\{i, j, k, l\} = \{0, 1, 2, 3\}$, the inward normal vector of the face $F_i$ is the vector $n_i$ such that $||n_i|| = 1$,
\[
n_i \cdot e_{j,k} = 0 \quad \text{and} \quad n_i \cdot e_{i,j} > 0
\]
Then, the inward equation of $F_i$ or the plane containing $F_i$ will be of the form $n_i \cdot (x - p_i) = 0$, where $p_i$ is a point on $F_i$. See Figure 8 for an illustration.

![Figure 8. Definition (5)](image1)

![Figure 9. Definition (5)](image2)

The incentre $I$ of $\Delta$ is the unique point $i$ such that
\[
n_0 \cdot (i - p_0) = n_i \cdot (i - p_1) = n_2 \cdot (i - p_2) = n_3 \cdot (i - p_3)
\]

(9)
i.e. equidistant to its faces from the interior. The common distance in (9) is called the inradius of Δ, denoted by \( r(\Delta) \). The sphere

\[
S^{in}(\Delta) := \{ x : ||x - i|| = r(\Delta) \}
\]

is called the insphere of Δ.

An interior angle bisector \( b_i^{in} \) of Δ at \( V_i \) is the locus of the point \( \chi/x \) satisfying

\[
n_j \cdot (x - p_j) = n_k \cdot (x - p_k) = n_l \cdot (x - p_l),
\]

such that \( \chi \) stays equidistant to the faces \( F_j, F_k \) and \( F_l \). See Figure 9 for an illustration.

We now prove the existence and uniqueness of the solution of (9) in order that incentre, inradius and insphere are well-defined, as well as that incentre can be characterized as the point of concurrence of interior angle bisectors. See Figure 10 for an illustration.

**Proposition 6. (Incentre as point of concurrence of interior angle bisectors)\** Let \( \Delta = [V_0, V_1, V_2, V_3] \) be a tetrahedron. Then, equation (9) has a unique solution \( i \).

Moreover, the incentre \( I \) of \( \Delta \) is the point of concurrence of its four interior angle bisectors.

**Figure 10. Proposition 6**

**Proof.** The equations captured by (9) are

\[
(n_i - n_0) \cdot i = n_i \cdot p_i - n_0 \cdot p_0 \quad \text{for } i = 1, 2, 3.
\]

It only suffices to prove that \( \{n_1 - n_0, n_2 - n_0, n_3 - n_0\} \) are linearly independent so that this system has a unique solution \( i \) as (4) does by arguing in a similar manner.
as in the proof of Proposition 4. To this end, suppose there are \(\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}\) such that
\[
\alpha_1(n_1 - n_0) + \alpha_2(n_2 - n_0) + \alpha_3(n_3 - n_0) = 0
\]
\[
\alpha_1n_1 + \alpha_2n_2 + \alpha_3n_3 = (\alpha_1 + \alpha_2 + \alpha_3)n_0
\]
Taking dot product with \(e_{0,i}\) for \(i \neq 0\), we have
\[
\alpha_i n_i \cdot e_{0,i} = (\alpha_1 + \alpha_2 + \alpha_3) n_0 \cdot e_{0,i} \quad (11)
\]
while taking dot product with \(e_{i,j}\) for \(i, j \neq 0\), we have
\[
\alpha_i n_i \cdot e_{i,j} + \alpha_j n_j \cdot e_{i,j} = 0
\]
\[
\alpha_i n_i \cdot e_{i,j} = \alpha_j n_j \cdot e_{i,j} \quad (12)
\]
By considering the signs of \(\alpha_1, \alpha_2, \alpha_3\), (11) and (12) are consistent only if
\[
\alpha_1 = \alpha_2 = \alpha_3 = 0
\]
and we have proved the claimed linear independence.

We have proved that \(\{n_1 - n_0, n_2 - n_0, n_3 - n_0\}\) are linearly independent, so that
\(\{n_0, n_1, n_2, n_3\}\) are affinely independent. Then, for each \(i\), \(\{n_j : j \neq i\}\) will also be affinely independent, so that the solutions of (10) form a line. Finally, it is trivial that the unique solution of (9) is the unique common solution of all those (10) for \(i = 0, 1, 2, 3\).

**Definition 7.** (Excentre and exterior angle bisector) Let \(\Delta = [V_0, V_1, V_2, V_3]\) be a tetrahedron. For \(\{i, j, k, l\} = \{0, 1, 2, 3\}\), the outward normal vector of the face \(F_i\) is the vector
\[
n'_i = -n_i
\]
Then, the outward equation of \(F_i\) or the plane containing \(F_i\) will be of the form
\[
n'_i \cdot (x - p_i) = 0, \text{ where } p_i \text{ is a point on } F_i.
\]
The excentre \(\mathcal{I}_i\) of \(\Delta\) opposite to \(V_i\) is the unique point \(i_s\) such that
\[
n'_i \cdot (i_s - p_i) = n_j \cdot (i_s - p_j) = n_k \cdot (i_s - p_k) = n_l \cdot (i_s - p_l).
\]
i.e. equidistant to its faces from the exterior. The common distance in (13) is called the exradius of \(\Delta\) opposite to \(V_i\), denoted by \(r_i(\Delta)\). The sphere
\[
S^e_x(\Delta) := \{x : ||x - i_s|| = r_i(\Delta)\}
\]
is called the exsphere of \(\Delta\) opposite to \(V_i\).

The exterior angle bisector \(b^e_{i,j}\) of \(\Delta\) at \(V_i\) opposite to \(V_j\) is the locus of the point \(\chi / x\) satisfying
\[
n'_j \cdot (x - p_j) = n_k \cdot (x - p_k) = n_i \cdot (x - p_i)
\]
such that \(\chi\) stays equidistant to the faces \(F_j, F_k, F_i\). Note that there are three exterior angle bisectors at each vertex. See Figure 11 for an illustration.
We now prove the existence and uniqueness of the solution of (13) in order that excentre, exradius and exsphere are well-defined, as well as that excentre can be characterized as the point of concurrence of interior and exterior angle bisectors. See Figure 12 for an illustration.
Proposition 8. (Excentre as point of concurrence of interior and exterior angle bisectors) Let \( \Delta = [V_0, V_1, V_2, V_3] \) be a tetrahedron. Then, for each \( i = 0, 1, 2, 3 \), equation (13) has a unique solution \( i \).

Moreover, the excentre \( I_i \) of \( \Delta \) opposite to \( V_i \) is the point of concurrence of the interior angle bisector \( b_i^{in} \) at \( V_i \) and the three exterior angle bisectors \( b_j^{ex}, b_k^{ex}, b_l^{ex} \) at \( V_j, V_k \) and \( V_l \) respectively opposite to \( V_i \).

Proof. Let \( \{i, j, k, l\} = \{0, 1, 2, 3\} \). The equations captured by (13) are

\[
\begin{align*}
(n_j + n_i) \cdot i &= n_j \cdot p_j + n_i \cdot p_i, \\
(n_k + n_i) \cdot i &= n_k \cdot p_k + n_i \cdot p_i, \\
(n_l + n_i) \cdot i &= n_l \cdot p_l + n_i \cdot p_i,
\end{align*}
\]

since \( n_i' = -n_i \). As with the case of incentre, it only suffices to prove that \( \{n_j + n_i, n_k + n_i, n_l + n_i\} \) are linearly independent so that this system has a unique solution \( i \). To this end, suppose there are \( \alpha_j, \alpha_k, \alpha_l \in \mathbb{R} \) such that

\[
\alpha_j(n_j + n_i) + \alpha_k(n_k + n_i) + \alpha_l(n_l + n_i) = 0
\]

Taking dot product with \( e_{i,j}, e_{i,k} \) and \( e_{i,l} \) respectively, we have

\[
\begin{align*}
(\alpha_j + \alpha_k + \alpha_l) n_i \cdot e_{i,j} &= -\alpha_j n_j \cdot e_{i,j}, \\
(\alpha_j + \alpha_k + \alpha_l) n_i \cdot e_{i,k} &= -\alpha_k n_k \cdot e_{i,k}, \\
(\alpha_j + \alpha_k + \alpha_l) n_i \cdot e_{i,l} &= -\alpha_l n_l \cdot e_{i,l},
\end{align*}
\]

which imply that either

\[
\alpha_j, \alpha_k, \alpha_l > 0 \quad \text{or} \quad \alpha_j, \alpha_k, \alpha_l < 0 \quad \text{or} \quad \alpha_j, \alpha_k, \alpha_l = 0
\]

But taking dot product with \( n_i \) in (15), with \( ||n_i|| = 1 \), we have

\[
\alpha_j + \alpha_k + \alpha_l = -\alpha_j n_j \cdot n_i - \alpha_k n_k \cdot n_i - \alpha_l n_l \cdot n_i
\]

If \( \alpha_j, \alpha_k, \alpha_l > 0 \), using the Cauchy-Schwarz inequality, with \( ||n_j|| = ||n_k|| = ||n_l|| = 1 \), then (16) would become

\[
\begin{align*}
\alpha_j, \alpha_k, \alpha_l &= \alpha_j(-n_j \cdot n_i) + \alpha_k(-n_k \cdot n_i) + \alpha_l(-n_l \cdot n_i) \\
&\leq \alpha_j ||n_j|| ||n_i|| + \alpha_k ||n_k|| ||n_i|| + \alpha_l ||n_l|| ||n_i|| \\
&= \alpha_j + \alpha_k + \alpha_l
\end{align*}
\]

Similarly, if \( \alpha_j, \alpha_k, \alpha_l < 0 \), then (16) would become

\[
\begin{align*}
\alpha_j, \alpha_k, \alpha_l &= (-\alpha_j) n_j \cdot n_i + (-\alpha_k) n_k \cdot n_i + (-\alpha_l) n_l \cdot n_i \\
&\geq (-\alpha_j)(-||n_j|| ||n_i||) + (-\alpha_k)(-||n_k|| ||n_i||) + (-\alpha_l)(-||n_l|| ||n_i||) \\
&= \alpha_j + \alpha_k + \alpha_l
\end{align*}
\]
However, both (17) and (18) forces \( n_j \cdot n_i = n_k \cdot n_i = n_l \cdot n_i = -1 \), which is impossible as it implies that \( n_i, n_j, n_k, n_l \) are parallel. Therefore, \( \alpha_j, \alpha_k, \alpha_l = 0 \), and we have proved the claimed linear independence.

We have proved that for \( \{i, j, k, l\} = \{0, 1, 2, 3\} \), \( \{n_j + n_i, n_k + n_i, n_l + n_i\} \) are linearly independent, so that \( \{-n_i, n_j, n_k, n_l\} \) are affinely independent. Then, for each \( i \), \( \{n_j, n_k, n_l\} \) will also be affinely independent, so that the solutions of (10) form a line. Also, for each \( i, j \), \( \{-n_j, n_k, n_l\} \) will also be affinely independent, so that the solutions of (14) form a line. Finally, by listing the following equations

\[
\begin{align*}
    b_{i \cdot n}^j : n_j \cdot (x - p_j) &= n_k \cdot (x - p_k) = n_l \cdot (x - p_l) \\
    b_{i \cdot n}^k : n_i' \cdot (x - p_i) &= n_k \cdot (x - p_k) = n_l \cdot (x - p_l) \\
    b_{i \cdot n}^l : n_k' \cdot (x - p_l) &= n_k \cdot (x - p_j) = n_l \cdot (x - p_l) \\
    b_{i \cdot n}^i : n_i' \cdot (x - p_i) &= n_k \cdot (x - p_j) = n_l \cdot (x - p_k)
\end{align*}
\]

it is trivial that the unique solution of (13) is the unique common solution of them.

\[\square\]

2.4. Orthocentre and Monge Point

**Definition 9.** (Orthocentre, altitude and orthocentric tetrahedron) Let \( \Delta = [V_0, V_1, V_2, V_3] \) be a tetrahedron. For \( i = 0, 1, 2, 3 \), the altitude \( h_i \) of \( \Delta \) from \( V_i \) is the line joining \( V_i \) and its projection \( H_i \) onto the face \( F_i \). \( H_i \) is also called the foot of altitude from \( V_i \).

If the four altitudes are concurrent, then the point of concurrence is called the orthocentre \( H \) of \( \Delta \), and \( \Delta \) is called an orthocentric tetrahedron.

**Definition 10.** (Monge point and Monge line) Let \( \Delta = [V_0, V_1, V_2, V_3] \) be a tetrahedron. The Monge point \( M \) of \( \Delta \) is defined as

\[
m = o + 2(g - o) = 2g - o \quad (19)
\]

where \( G \) and \( O \) are the centroid and the circumcentre of \( \Delta \) respectively.

For \( i = 0, 1, 2, 3 \), let \( P_i \) be the mid-point of \( [H_i, M_i] \), where \( M_i \) abbreviates the orthocentre \( M(F_i) \) of the face \( F_i \). Then, a Monge line \( m \) of \( \Delta \) is the line passing through \( P_i \) and perpendicular to \( F_i \) (cf. [13] where the same line has also been introduced without this name). See Figure 13 for an illustration.
We now prove that Monge point can be characterized as the point of concurrence of Monge lines. See Figure 14 for an illustration.

**Proposition 11.** *(Monge point as point of concurrence of Monge lines)* The Monge point $\mathcal{M}$ of a tetrahedron $\Delta = [V_0, V_1, V_2, V_3]$ is the point of concurrence of its four Monge lines.
Proof. Let $i = 0, 1, 2, 3$. It is well-known that the orthocentre of $F_i$ is given as
\[ m_i = 3g_i - 2o_i \] (20)
so that the Monge point of $\Delta$
\[ m = 2g - o = \frac{1}{2} (v_i + 3g_i) - o = \frac{1}{2} (v_i + m_i + 2o_i) - o = \frac{1}{2} (v_i + m_i) + (o_i - o) \] (21)
Let $P'_i$ be the midpoint of $[V_i, M_i]$. By Proposition 4, the last term $o_i - o$ in (21) is a vector perpendicular to $F_i$, thus (21) shows that $M$ lies on the line $l_i$ passing through $P'_i$ and perpendicular to $F_i$. But by applying the intercept theorem in $\Delta M_i V_i H_i$, $l_i$ actually hits $P_i$, showing that $M$ lies on $m_i$.

Moreover, the Monge lines cannot intersect at more points, otherwise, they cannot be distinct lines since two points determine a line. \qed

**Corollary 12.** (Monge point of orthocentric tetrahedron) In an orthocentric tetrahedron $\Delta = [V_0, V_1, V_2, V_3]$, the foot of altitude $H_i$ coincides with the orthocentre $M_i$ of the face $F_i$, and the Monge point $M$ of $\Delta$ coincides with the orthocentre $H$ of $\Delta$.

Proof. Let $\{i, j, k, l\} = \{0, 1, 2, 3\}$. Since the orthocentre $H$ of the orthocentric tetrahedron $\Delta$ is the intersection of the altitudes $h_i$ and $h_j$, we have
\[ t_i v_i + (1 - t_i) h_i = h = t_j v_j + (1 - t_j) h_j \]
\[ v_j - h_i = t_i (v_i - h_i) + (1 - t_j) (v_j - h_j) \]
\[ \perp F_i \]
\[ \perp F_j \]
for some $t_i, t_j \in \mathbb{R}$. But as $(v_i - h_i) \cdot e_{kl} = 0$ and $(v_j - h_j) \cdot e_{kl} = 0$, we get
\[ (v_j - h_i) \cdot e_{kl} = t_i \cdot 0 + (1 - t_j) \cdot 0 = 0 \]
which is saying that the line joining $V_j$ and $H_i$ is perpendicular to the edge $E_{k,l}$ of the face $F_i$. Hence, $H_i$ is the orthocentre $M_i$ of $F_i$.

Once $H_i = M_i$, from Figure 13, the Monge line $m_i$ will coincide with the altitude $h_i$. As a result, the point of concurrence of the Monge lines (i.e. the Monge point according to Proposition 11) coincides with the point of concurrence of the altitudes (i.e. orthocentre). \qed

In view of Corollary 12, Monge point is a perfect generalization of orthocentre. And as the altitudes of a triangle must be concurrent so that every triangle can be regarded as being orthocentric, the orthocentre of a triangle can be regarded as the Monge point of the triangle. This justifies denoting the orthocentre of the face $F_i$ by $M_i$ in Definition 10.
3. Quasi-orthocentres and Euler Line

This section will show one of the major discoveries of this paper: the *quasi-orthocentres*, which was inspired by the line characterization of Monge point in Section 2.4.

It is immediate from (19) in Definition 10 that the Monge point $M$ of a tetrahedron lies on the Euler line $E$ of the tetrahedron (i.e. the line joining the centroid $G$ and the circumcentre $O$, which is well-defined only when $G \neq O$). Similarly, (20) also shows that the orthocentre / Monge point $M_i$ of a face lies on the Euler line $E_i$ of the face (i.e. the line joining the centroid $G_i$ and the circumcentre $O_i$ of the face, which is well-defined only when $G_i \neq O_i$).

Proposition 11 in Section 2.4 is therefore demonstrating the following nice geometric feature of the Monge point of a tetrahedron: it is the point of concurrence of lines (i) parallel to the altitudes and (ii) emerging from specific points of division of the segments joining the feet of altitude and certain facial centres lying on the Euler lines of the faces. From the proof, it seems that the scaling factor 2 in (19) was so carefully chosen to facilitate such feature, but we will discover a family of tetrahedron centres that possess the same feature.

Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron, and let $i = 0, 1, 2, 3$. Let $\chi_i$ be a point on the Euler line $E_i$ of the face $F_i$ of $\Delta$ given as

$$ x_i = o_i + r(g_i - o_i) = rg_i + (1 - r)o_i, $$

where $r \in \mathbb{R}\{0\}$ is a constant. Also let $Y$ be a point on the Euler line $E$ of $\Delta$ given as

$$ y = sg + (1 - s)o, $$

where $s \in \mathbb{R}$ is a constant. Then,

$$ y = \frac{s}{4}(v_i + 3g_i) + (1 - s)o $$

$$ = \frac{s}{4}\left(v_i + \frac{3}{r}x_i - \frac{3(1-r)}{r}o_i\right) + (1 - s)o $$

$$ = \frac{s}{4}v_i + \frac{3s}{4r}x_i - \frac{3s(1-r)}{4r}o_i + (1 - s)o \quad (22) $$

For (22) to mean that $Y$ lies on a line parallel to the altitude $h_i$ and passing through a point on the line joining $V_i$ and $\chi_i$, it requires first of all that

$$ \frac{s}{4} + \frac{3s}{4r} = 1 $$

$$ s = \frac{4r}{r+3} \quad \text{and} \quad r \neq -3 \quad (23) $$
Letting \( r = \frac{3}{2+k} \) in (23), where \( k \in \mathbb{R}\setminus\{-2,-3\} \), we have

\[
s = \frac{4 \cdot \frac{3}{2+k}}{2+k} + 3 = \frac{4}{3+k}
\] (24)

Substituting (24) into (22), we have

\[
y = \frac{1}{3+k}v_i + \frac{2+k}{3+k}x_i + \frac{1-k}{3+k}(o_i - o)
\] (25)

where the last term is just a vector parallel to \( F_i \) according to Proposition 4.

Hence, we can now define:

**Definition 13.** *(Quasi-orthocentre and quasi-altitude)* Fix any \( k \neq -2, -3 \). Let \( \Delta = [V_0, V_1, V_2, V_3] \) be a tetrahedron. The \( k \)-quasi-orthocentre \( Q_k \) of \( \Delta \) is defined as

\[
q_k = o + \frac{4}{3+k}(g - o) = \frac{4}{3+k}g - \frac{1-k}{3+k}o,
\] (26)

where \( G \) and \( O \) are the centroid and the circumcentre of \( \Delta \) respectively.

The \( k \)-quasi-orthocentre \( Q_{k,i} \) of the face \( F_i \) of \( \Delta \) is defined as

\[
q_{k,i} = o_i + \frac{3}{2+k}(g_i - o_i) = \frac{3}{2+k}g_i - \frac{1-k}{2+k}o_i
\]

where \( G_i \) and \( O_i \) are the centroid and the circumcentre of \( F_i \) respectively.

For \( i = 0, 1, 2, 3 \), let \( P_i \) be the point on the line joining \( H_i \) and \( Q_{k,i} \) such that \( H_iP_i : P_iQ_{k,i} = (2+k) : 1 \). Then, a \( k \)-quasi-altitude \( q_{k,i} \) of \( \Delta \) is the line passing through \( P_i \) and perpendicular to \( F_i \). See Figure 15 for an illustration.
By (25), we have actually proved that the $k$-quasi-orthocentre of a tetrahedron is the point of concurrence of its four $k$-quasi-altitudes. See Figure 16 for an illustration.

**Proposition 14.** *(Quasi-orthocentre as point of concurrence of quasi-altitudes) Fix any $k \neq -2, -3$. The $k$-quasi-orthocentre $Q_k$ of a tetrahedron $\Delta = [V_0, V_1, V_2, V_3]$ is the point of concurrence of its four $k$-quasiaaltitudes.*
Note that Monge point $M$ and twelve-point centre $N$ of tetrahedra are members of the family of quasiorthocentres $Q_k$:

$$m = 2g - o = q_{-1} \quad \text{and} \quad n = \frac{4}{3}g - \frac{1}{3}o = q_0$$

For twelve-point centre, see [2], [3], [4] and [18]. Also, their triangle counterparts — orthocentre $M/H$ and nine-point centre $N$ — are also quasi-orthocentres $Q_{k,i}$ of faces of tetrahedra:

$$m/h = 3g - 2o = q_{-1} \quad \text{and} \quad n = \frac{3}{2}g - \frac{1}{2}o = q_0$$

Hence, Monge point and twelve-point centre of tetrahedra share the common geometric feature of being the point of concurrence of altitude-like special lines derived from their triangle counterparts.

4. Antimedial Circumcentre and Orthic Inexcentre

This section will show the other two major discoveries of this paper: the antimedial circumcentre and the orthic inexcentre, as two new generalizations of orthocentre of triangles. As shown in Section 2.4, Monge point of tetrahedra generalizes orthocentre of triangles as the latter is regarded as the point of concurrence of altitudes. But using altitudes is not the only way to characterize orthocentre of triangles (cf. [10]).

Indeed, a quick proof for high school students of the fact that the three altitudes of a triangle concur at the orthocentre of the triangle is often through considering its antimedial triangle, whereby the altitudes of the former become the perpendicular bisectors of the latter. Figures 17 and 18 show that the orthocentre $H(\Delta)$ of $\Delta = \Delta ABC$ is exactly the circumcentre $O(\Delta')$ of its antimedial triangle $\Delta' = \Delta'A'B'C'$. This motivates us to introduce antimedial tetrahedron in a natural manner in Section 4.2, and its circumcentre will be one generalization of orthocentre. In fact, [18] has proved that the antimedial circumcentre, without using this name, lies on the Euler line. Our work will be re-proving its tetrahedron version with simpler presentation.

The characterization of the orthocentre of a triangle as the incentre or an excentre of its orthic triangle may be less well-known to high school students. Figures 19 and 20 show that the orthocentre $H(\Delta)$ of $\Delta = \Delta ABC$ is exactly the incentre $I(\Delta')$ (when $\Delta$ is acute-angled) or the excentre $I_A(\Delta')$ (when $\Delta$ is obtuse-angled at $A$) of its orthic triangle $\Delta' = \Delta'A'B'C'$. This inspires us to construct orthic tetrahedron in Section 4.3, and its incentre or excentre will be another generalization of orthocentre.
The two most natural kinds of orthic tetrahedron have already been studied in [5], but their vertices are confined to the planes containing the faces of the original tetrahedron. One could definitely consider their incentres or excentres as analogues of orthocentre of triangles. But we found that none of these could carry any good properties of orthic triangles or orthocentre of triangles over to tetrahedra. We will think out of the box to construct a new kind of orthic tetrahedron, whose vertices need not be restricted as suffered by the ordinary orthic tetrahedra.

Actually, we will prove that the following well-known properties of orthocentre of triangles can carry over to tetrahedra through antimedial circumcentre:

i The orthocentre \( H \) of a triangle is collinear with its centroid \( G \) and circumcentre \( O \).

ii The orthocentre \( h \) of a triangle \([V_0, V_1, V_2]\) can be expressed as

\[
o + (g_0 - o) + (g_1 - o) + (g_2 - o)
\]
where \( o \) is its circumcentre and \( g_i := \frac{1}{2}(v_j + v_k) \) for \( \{i,j,k\} = \{0,1,2\} \) is treated as the centroid of the edge \( E_{j,k} \).

We will also prove that the following well-known properties of orthocentre of triangles can carry over to tetrahedra through orthic in/excentre:

i The orthocentre \( H \) of a triangle is collinear with its circumcentre \( O \) and twenty-fifth Kimberling centre \( \chi_{25} \).

ii The orthocentre \( h \) of a triangle \( [V_0, V_1, V_2] \) can be expressed as
\[
o + (o_0 - o) + (o_1 - o) + (o_2 - o)
\]
where \( o \) is its circumcentre and \( o_i := \frac{1}{2}(v_j + v_k) \) for \( \{i,j,k\} = \{0,1,2\} \) is treated as the circumcentre of the edge \( E_{j,k} \).

Both Sections 4.2 and 4.3 will be highly related to homothety. Therefore, before them, we will prove a necessary and sufficient condition for homothety between two tetrahedra in Section 4.1.

4.1. Homothetic Tetrahedra

**Definition 15. (Homothety)** A homothetic transformation \( T \) is a function \( T : \mathbb{R}^3 \rightarrow \mathbb{R}^3 \) of the form
\[
T(u) = z + t(u - z) = tu + (1 - t)z \quad \text{for} \quad u \in \mathbb{R}^3
\]
where \( z \in \mathbb{R}^3 \) is called the homothetic centre and \( t \in \mathbb{R}\setminus \{0\} \) is called the homothetic ratio.

Two tetrahedra \( \Delta \) and \( \Delta' \) are said to be homothetic if one of them can be obtained from the other through a homothetic transformation, i.e. there exists a homothetic transformation \( T \) such that \( T(\Delta) = \Delta' \). The homothetic centre will be denoted by \( Z(\Delta, \Delta') \).

There is no harm to say that a translation transformation
\[
u \mapsto u + b
\]
is a homothetic transformation with homothetic centre \( z = \infty \) and homothetic ratio \( t = 1 \).

Note that from (27) and (28) that \( z, u \) and \( T(u) \) must be collinear.

To justify that homothety between two tetrahedra is well-defined, we need the following lemma:

**Lemma 16. (Triangle and tetrahedron under homothetic transformation)** Let \( \Delta = [V_0, V_1, V_2, V_3] \) be a tetrahedron and \( T \) be a homothetic transformation. Then,

(a) \( \{T(v_0), T(v_1), T(v_2), T(v_3)\} \) are affinely independent,
Proof. Under homothetic transformation $T$ of the form (27) or (28), we have

$$T(v_i) - T(v_0) = t(v_i - v_0) \quad \text{for } i = 1, 2, 3.$$  

Suppose there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\alpha_1(T(v_1) - T(v_0)) + \alpha_2(T(v_2) - T(v_0)) + \alpha_3(T(v_3) - T(v_0)) = 0$$

$$t\alpha_1(v_1 - v_0) + t\alpha_2(v_2 - v_0) + t\alpha_3(v_3 - v_0) = 0$$

$$\alpha_1(v_1 - v_0) + \alpha_2(v_2 - v_0) + \alpha_3(v_3 - v_0) = 0$$

Since $\{v_1 - v_0, v_2 - v_0, v_3 - v_0\}$ are linearly independent, we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore, the transformed vertices $\{T(v_0), T(v_1), T(v_2), T(v_3)\}$ are affinely independent too, hence proving (a).

As mentioned in Section 1.3, any point $x$ in the tetrahedron can be represented as a convex combination as in (1). If $T$ takes the form of (27), where $t \neq 1$, then

$$T(x) = t(\alpha_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3) + (1 - t)z$$

$$= \lambda_0(t v_0 + (1 - t)z) + \lambda_1(t v_1 + (1 - t)z) + \lambda_2(t v_2 + (1 - t)z)$$

$$+ \lambda_3(t v_3 + (1 - t)z)$$

$$= \lambda_0 T(v_0) + \lambda_1 T(v_1) + \lambda_2 T(v_2) + \lambda_3 T(v_3).$$

If $T$ takes the form of (28), then

$$T(x) = \lambda_0 v_0 + \lambda_1 v_1 + \lambda_2 v_2 + \lambda_3 v_3 + b$$

$$= \lambda_0(v_0 + b) + \lambda_1(v_1 + b) + \lambda_2(v_2 + b) + \lambda_3(v_3 + b)$$

$$= \lambda_0 T(v_0) + \lambda_1 T(v_1) + \lambda_2 T(v_2) + \lambda_3 T(v_3).$$

In both cases, $T(x) = \lambda_0 T(v_0) + \lambda_1 T(v_1) + \lambda_2 T(v_2) + \lambda_3 T(v_3)$, which shows that as $\lambda_0, \lambda_1, \lambda_2, \lambda_3$ run through the condition in (1) so that $x$ runs through $\Delta$, $T(x)$ will run through every point the tetrahedron $\{T(v_0), T(v_1), T(v_2), T(v_3)\}$ in a one-to-one correspondence manner. More precisely, homothetic transformations preserve convex combination. Hence, (b), (c) and (d) follow. \qed

From Lemma 16, a tetrahedron is transformed to another tetrahedron through a homothetic transformation, so homothety between two tetrahedra is well-defined.

It is easy to check that homothetic tetrahedra have parallel edges and faces:

**Lemma 17.** (Edges and faces of homothetic tetrahedra) Two homothetic tetrahedra have parallel corresponding edges and faces.

Proof. Let $T$ be as in (27) or (28), $\Delta = [V_0, V_1, V_2, V_3]$ and $\Delta' = [V'_0, V'_1, V'_2, V'_3] = T(\Delta) = \Delta'$. Let $\{i, j, k, l\} = \{0, 1, 2, 3\}$, then $e'_{i,j} = v'_i - v'_j = T(v_i) - T(v_j) =
tv_1 - tv_j = te_{i,j}, showing that E_{i,j}||E'_{i,j}. Thus, corresponding edges of \( \Delta \) and \( \Delta' \) are parallel.

If \( n_k \) is a normal vector of the face \( F_k \), then \( n_k \cdot e_{i,j} = 0 \), and then \( n_k \cdot e'_{i,j} = n_k \cdot te_{i,j} = tn_k \cdot e_{i,j} = 0 \), showing that \( n_k \) is also a normal vector of the face \( F'_k \), and \( F_k||F'_k \). Thus, corresponding faces of \( \Delta \) and \( \Delta' \) are parallel.

How about the converse of Lemma 17? The next two lemmas will prove it:

**Lemma 18.** *(Tetrahedra with parallel corresponding faces) Two tetrahedra with parallel corresponding faces have parallel corresponding edges.*

**Proof.** Consider \( \Delta = [V_0, V_1, V_2, V_3] \) and \( \Delta' = [V'_0, V'_1, V'_2, V'_3] \) where \( F_i||F'_i \) for \( i = 0, 1, 2, 3 \). Let \( \{i, j, k, l\} = \{0, 1, 2, 3\} \), and consider the direction vectors \( e_{i,j} \) and \( e'_{i,j} \).

If \( n_k \) and \( n_l \) are the respective common normal vectors of \( F_k \) and \( F'_k \) and of \( F_l \) and \( F'_l \), then we have

\[
\begin{align*}
n_k \cdot e_{i,j} &= 0 \\
n_i \cdot e_{i,j} &= 0
\end{align*}
\]

and

\[
\begin{align*}
n_k \cdot e'_{i,j} &= 0 \\
n_i \cdot e'_{i,j} &= 0
\end{align*}
\]

as \( E_{i,j} \) and \( E'_{i,j} \) are the respective intersections of \( F_k \) and \( F'_k \) and of \( F_l \) and \( F'_l \). This means that both \( e_{i,j} \) and \( e'_{i,j} \) are solutions to the system of linear equations

\[
(n_k \quad n_l)^T x = 0 \tag{29}
\]

But \( n_k \) and \( n_l \) have to be linearly independent, because otherwise \( F_k \) and \( F'_k \) would be parallel, therefore, the solution space of (29) is one-dimensional, i.e. \( E_{i,j}||E'_{i,j} \).

Hence, corresponding edges of \( \Delta \) and \( \Delta' \) are parallel.

**Lemma 19.** *(Tetrahedra with parallel edges) Two tetrahedra with parallel corresponding edges have homothetic.*

**Proof.** Consider \( \Delta = [V_0, V_1, V_2, V_3] \) and \( \Delta' = [V'_0, V'_1, V'_2, V'_3] \) where \( E_{i,j}||E'_{i,j} \) for \( i, j = 0, 1, 2, 3 \) and \( i \neq j \). Let \( e'_{0,1} = te_{0,1} \), where \( t \neq 0 \).

If \( t \neq 1 \), then from

\[
\begin{align*}
e'_{0,1} &= te_{0,1} \\
v'_0 - v'_1 &= t(v_0 - v_1)
\end{align*}
\]

\[
\begin{align*}
\frac{1}{1-t}v'_0 + \frac{-t}{1-t}v_0 &= \frac{1}{1-t}v'_1 + \frac{-t}{1-t}v_1
\end{align*}
\]
we see that the lines $V_0 V'_0$ and $V_1 V'_1$ intersect at $Z/z$

$$z := \frac{1}{1 - t} v'_0 + \frac{-t}{1 - t} v_0 = \frac{1}{1 - t} v'_1 + \frac{-t}{1 - t} v_1$$  

(30)

We will show that the required homothetic transformation is

$$T(u) = z + t(u - z) \quad \text{for } u \in \mathbb{R}^3$$

Indeed, by Lemma 33, it only suffices to show that it satisfies $T(v_i) = v'_i$ for $i = 0, 1, 2, 3$. To this end, let $i = 0, 1$ first, then from (30),

$$z = \frac{1}{1 - t} v'_i + \frac{-t}{1 - t} v_i$$

$$v'_i = (1 - t)z + tv_i = T(v_i)$$  

(31)

**Figure 21.** Lemma 19  
**Figure 22.** Lemma 19

For $i = 2, 3$, consider the line

$$l_0 : x_0(r) = v'_0 + r(v_0 - v_i) \quad \text{for } r \in \mathbb{R}$$  

(32)

through $V'_0$ and parallel to $[V_0, V_i]$, and the line

$$l_1 : x_1(s) = v'_1 + s(v_1 - v_0) \quad \text{for } s \in \mathbb{R}$$  

(33)

through $V'_1$ and parallel to $[V_1, V_i]$. Refer to Figure 21. Since $E_{0,i} \parallel E'_{0,i}$ and $E_{1,i} \parallel E'_{1,i}$, $l_0$ and $l_1$ intersect at $V'_i$. Setting $x_0(r) = x_1(s)$, with (31), we have

$$(1 - t)z + tv_0 + r(v_0 - v_1) = (1 - t)z + tv_1 + s(v_1 - v_i)$$

$$(t + r)(v_0 - v_i) = (t + s)(v_1 - v_i)$$

But $\{v_0 - v_i, v_1 - v_i\}$ are linearly independent, so $t + r = t + s = 0$ or $r = s = -t$. Substituting into (32) and using (31), we have

$$v'_i = (1 - t)z + tv_0 - t(v_0 - v_i) = T(v_i),$$
and we are done.

If \( t = 1 \), then from

\[
\begin{align*}
\mathbf{e}_{0,1}' &= \mathbf{e}_{0,1} \\
\mathbf{v}_0' - \mathbf{v}_0 &= \mathbf{v}_1' - \mathbf{v}_1,
\end{align*}
\]

we can construct a homothetic (actually translation) transformation

\[
T(\mathbf{u}) = \mathbf{u} + \mathbf{b} \quad \text{for } \mathbf{u} \in \mathbb{R}^3,
\]

where

\[
\mathbf{b} := \mathbf{v}_0' - \mathbf{v}_0 = \mathbf{v}_1' - \mathbf{v}_1,
\]

and show that this is the required one. To this end, let \( i = 0,1 \) first, then from (34),

\[
\begin{align*}
\mathbf{b} &= \mathbf{v}_i' - \mathbf{v}_i \\
\mathbf{v}_i' &= \mathbf{v}_i + \mathbf{b} = T(\mathbf{v}_i)
\end{align*}
\]

For \( i = 2,3 \), consider the lines (32) and (33). Refer to Figure 22. Again, since \( E_{0,i}||E_{0,i}' \) and \( E_{1,i}||E_{1,i}' \), \( l_0 \) and \( l_1 \) intersect at \( V_i' \). Setting \( x_0(r) = x_1(s) \), with (31), we have

\[
\begin{align*}
\mathbf{v}_0 + \mathbf{b} + r(\mathbf{v}_0 - \mathbf{v}_i) &= \mathbf{v}_1 + \mathbf{b} + s(\mathbf{v}_1 - \mathbf{v}_i) \\
(1 + r)(\mathbf{v}_0 - \mathbf{v}_i) &= (1 + s)(\mathbf{v}_1 - \mathbf{v}_i),
\end{align*}
\]

But \( \{\mathbf{v}_0 - \mathbf{v}_i, \mathbf{v}_1 - \mathbf{v}_i\} \) are linearly independent, so \( 1 + r = 1 + s = 0 \) or \( r = s = -1 \).

Substituting into (32) and using (35), we have

\[
\begin{align*}
\mathbf{v}_i' &= \mathbf{v}_0 + \mathbf{b} - (\mathbf{v}_0 - \mathbf{v}_i) = T(\mathbf{v}_i)
\end{align*}
\]

and we are done. \( \square \)

Be aware that Lemma 19 does not work for polyhedra other than tetrahedra as the above proof relies on that in a tetrahedron any two vertices are connected by an edge. One counterexample is triangular prism in which changing the height produces parallel corresponding faces and edges but does not produce a homothetic triangular prism.

Combining Lemma 17, 18 and 19, we obtain:

**Theorem 20.** (Necessary and sufficient condition for homothetic tetrahedra) Two tetrahedra are homothetic if and only if they have parallel corresponding faces.

### 4.2. Circumcentre of Antimedial Tetrahedron

Recall from Proposition 4 that circumcentre must uniquely exist in any tetrahedron. For the sake of brevity, from this point on, we assume, without any loss of generality,
that the circumcentre $O$ of a tetrahedron $\Delta = [V_0, V_1, V_2, V_3]$ is the origin. Thus, in particular,

$$||v_0|| = ||v_1|| = ||v_2|| = ||v_3|| = R,$$

where $R$ is the circumradius of $\Delta$.

**Definition 21.** *(Antimedial tetrahedron and antimedial circumcentre)* The antimedial tetrahedron $\Delta_{am} = [V'_0, V'_1, V'_2, V'_3]$ of a tetrahedron $\Delta = [V_0, V_1, V_2, V_3]$ is the tetrahedron whose faces $F'_0, F'_1, F'_2, F'_3$ satisfy

$$V_i \in F'_i \quad \text{and} \quad F'_i || F_i \quad \text{for} \quad i = 0, 1, 2, 3.$$

The antimedial circumcentre $J/j$ of a tetrahedron $\Delta$ is the circumcentre of its antimedial tetrahedron $\Delta_{am}$, i.e.

$$J(\Delta) = O(\Delta_{am}).$$

We now prove the important properties, one algebraic and one geometric, of orthocentre of triangles that are carried over through generalizing as antimedial circumcentre of tetrahedra. See Figure 23 for an illustration.

**Theorem 22.** *(Antimedial circumcentre)* Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. If its circumcentre $O$ is the origin, then its antimedial circumcentre $J$ is given by

$$j = 4g = v_0 + v_1 + v_2 + v_3 = g_0 + g_1 + g_2 + g_3 \quad (36)$$

where $G_i$ denotes the centroid of the face $F_i$.

Moreover, $J$ lies on the Euler line $E$ of $\Delta$.

![Figure 23. Theorem 22](image)
Proof. Let $V'_i/V'_i$ be a point such that
\[ g = \frac{3}{4}v_i + \frac{1}{4}v'_i \]  
(i.e. the centroid $G$ of $\Delta$ divides $[V_i, V'_i]$ internally in $V_iG : GV'_i = 1 : 3$, or
\[ v'_i = 4g - 3v_i = j - 3v_i. \]
Recalling that \{v_0, v_1, v_2, v_3\} are affinely independent, then by noting that $v'_i - v'_j = -3(v_i - v_j)$, so are $\{v'_0, v'_1, v'_2, v'_3\}$. We shall show that
\[ \Delta' := [V'_0, V'_1, V'_2, V'_3] = \Delta_{am} \]
Let $\{i, j, k, l\} = \{0, 1, 2, 3\}$. Note that
\[ v'_j + v'_k + v'_l = 12g - 3(v_j + v_k + v_l) \]
\[ = 12g - 3(4g - v_i) \]
\[ v_i = \frac{1}{3}(v'_j + v'_k + v'_l) \]
which shows that $V_i$ is the centroid $G(F'_i)$ of $F'_i$ and lies on $F'_i$. Moreover, if $n_i$ is a normal vector of $F'_i$, i.e. $n_i \cdot (v_j - v_k) = 0$, then
\[ n_i \cdot (v'_j - v'_k) = -3n_i \cdot (v_j - v_k) = 0, \]
i.e. $n_i$ is also a normal vector of $F'_i$, so $F_i || F'_i$. Hence $\Delta' = \Delta_{am}$.
Checking that
\[ ||j - v'_i|| = ||3v_i|| = 3R, \]
where $R$ is the circumradius of $\Delta$, so (36) provides $J(\Delta) = O(\Delta_{am})$.
Moreover, (37) also means that all $[V_i, W_i]$ concur at $G$, so according to the proof of Lemma 19, $G = Z(\Delta, \Delta_{am})$. Therefore, $J(\Delta) = O(\Delta_{am})$ lies on the line joining $O(\Delta)$ and $G$, i.e. the Euler line $E$ of $\Delta$. \hfill $\square$

Note that if the circumcentre $O$ is not assumed to be the origin, then (36) should be modified as
\[ j - o = 4(g - o) = \left[(v_0 - o) + (v_1 - o) + (v_2 - o) + (v_3 - o)\right] \]
\[ = (g_0 - o) + (g_1 - o) + (g_2 - o) + (g_3 - o) \]
\[ j = 4g - 3o = v_0 + v_1 + v_2 + v_3 - 3o = g_0 + g_1 + g_2 + g_3 - 3o \]  
(38)

4.3. Inexcentre of Orthic Tetrahedron

Again, for the sake of brevity, the circumcentre $O$ of a tetrahedron
\[ \Delta = [V_0, V_1, V_2, V_3] \]
will be assumed to be at the origin without any loss of generality. Thus, in particular,
\[ ||v_0|| = ||v_1|| = ||v_2|| = ||v_3|| = R, \]
where \( R \) is the circumradius of \( \Delta \).

Figures 24 (when \( \Delta ABC \) is acute-angled) and 25 (when \( \Delta ABC \) is obtuse-angled) recall how an edge of an orthic triangle may be obtained by a semi-circle. This is the underlying idea of our construction of orthic tetrahedron.

\[
v_i \cdot (x - v_i) = 0 \quad \text{or} \quad v_i \cdot x = R^2
\] (39)

where \( R \) is the circumradius of \( \Delta \). See Figure 26 for an illustration.
The tangential tetrahedron $\Delta_{tg}$ of $\Delta$ is the tetrahedron enclosed by its four tangential planes. Note that the position of the circumcentre $O$ of $\Delta$ determines one of the following configurations:

(i) Acute-angled case: If $O(\Delta)$ lies inside $\Delta$, then $\Delta_{tg}$ touches $\Delta$ at all the vertices of $\Delta$, and

$$O(\Delta) = T(\Delta_{tg})$$

as shown in Figure 27.

(ii) Obtuse-angled case: If $O(\Delta)$ lies outside $\Delta$ and not on any of (the planes containing) a face of $\Delta$, then $\Delta_{tg}$ touches $\Delta$ only at the vertex $V_i$ opposite to $O(\Delta)$, and

$$O(\Delta) = I_i(\Delta_{tg})$$

as shown in Figure 28. The vertex $V_i$ is called the obtuse vertex of $\Delta$.

(iii) Right-angled case: If $O(\Delta)$ lies on (the plane containing) a face of $\Delta$, then $\Delta_{tg}$ cannot be formed as shown in Figure 29, and tangential tetrahedron would be undefined.
Beware the terminologies *acute-angled*, *obtuse-angled* and *right-angled*, which classify triangles by angle size, may not mean anything about angle size in a tetrahedron. They are borrowed from the two-dimensional scenario only to mean those in Definition 23.

**Definition 24.** *(Orthic plane)* Consider a tetrahedron $\Delta = [V_0, V_1, V_2, V_3]$. Let $\{i, j, k, l\} = \{0, 1, 2, 3\}$. Construct a sphere $S_i$ with centre $O_i$ and radius $R_i = O_iV_j = O_iV_k = O_iV_l$, where $O_i$ is the circumcentre of face $F_i$. Denote by $W_{ij}^i$ the reflection of $V_j$ across the projection of $O_i$ onto the (extended) edge $E_{i,j}$. Note that $W_{ij}^i$ is simply the intersection of the sphere $S_i$ and the (extended) edge $E_{i,j}$ other
than \( V_j \), unless \( E_{i,j} \perp [V_j, O_i] \) in which \( W_j^i = V_j \). Likewise, the points \( W_k^i \) and \( W_l^i \) are defined.

The orthic plane \( U_i \) of \( \Delta \) (corresponding to \( V_i \)) is the plane containing \( W_j^i, W_k^i \) and \( W_l^i \). See Figures 30 (acute-angled case) and 31 (obtuse-angled case) for illustrations. Note that orthic plane is undefined if \( O_i \) coincides with the circumcentre \( O \) of \( \Delta \) as \( W_j^i = W_k^i = W_l^i \) (right-angled case).

**Figure 30. Definition 24**

**Figure 31. Definition 24**

**Lemma 25. (Orthic plane parallel to tangent plane)**

For a tetrahedron \( \Delta = [V_0, V_1, V_2, V_3] \), the tangential plane \( T_i \) is parallel to the orthic plane \( U_i \).

**Proof.** (Analytic approach) Refer to Figure 32. Let \( \{i, j, k, l\} = \{0, 1, 2, 3\} \). Since \( v_i \perp T_i \) according to Definition 23, the statement is equivalent to proving that \( v_i \perp U_i \), then one needs to find out why

\[
(w_j^i - w_k^i) \cdot v_i = 0 \quad \text{and} \quad (w_j^i - w_l^i) \cdot v_i = 0 \tag{40}
\]

To this end, we shall compute

\[
w_j^i \cdot v_i \cdot w_k^i \cdot v_i \quad \text{and} \quad w_l^i \cdot v_i \tag{41}
\]

Recall that the circumradius \( R \) of \( \Delta \) satisfies \( R = ||v_i|| = ||v_j|| = ||v_k|| = ||v_l|| \). Also recall that the circumradius \( R_i \) of \( F_i \) satisfies

\[
R_i = ||v_j - o_i|| = ||v_k - o_i|| = ||v_l - o_i||, \tag{42}
\]

from which we get

\[
R_i^2 = ||v_j||^2 - 2v_j \cdot o_i + ||o_i||^2 = R^2 - 2v_j \cdot o_i + R^2 - R_i^2 \quad (\because o_i \perp F_i \text{ from Proposition 4})
\]

\[
R^2 - v_j \cdot o_i = R_i^2 \tag{43}
\]
Since \( \mathbf{w}_j^i \) lies on the (extended) edge \( E_{i,j} \), we let
\[
\mathbf{w}_j^i = t\mathbf{v}_i + (1 - t)\mathbf{v}_j \quad \text{for some } t \in \mathbb{R}\{0\}
\]  
(44)

\[\text{Figure 32. Lemma 25}\]

By the definition of \( W_j^i \),
\[
||\mathbf{w}_j^i - \mathbf{o}_i|| = R_i
\]
\[||t(\mathbf{v}_i - \mathbf{v}_j) + (\mathbf{v}_j - \mathbf{o}_i)|| = R_i \quad \text{(by (44))}\]
\[t^2||\mathbf{v}_i - \mathbf{v}_j||^2 + 2t(\mathbf{v}_i - \mathbf{v}_j) \cdot (\mathbf{v}_j - \mathbf{o}_i) + ||\mathbf{v}_j - \mathbf{o}_i||^2 = R_i^2\]
\[t^2||\mathbf{v}_i - \mathbf{v}_j||^2 + 2(\mathbf{v}_i - \mathbf{v}_j) \cdot (\mathbf{v}_j - \mathbf{o}_i) = 0 \quad \text{(by (42) and } t \neq 0)\]

\[
t = \frac{2(\mathbf{v}_j - \mathbf{v}_i) \cdot (\mathbf{v}_j - \mathbf{o}_i)}{||\mathbf{v}_i - \mathbf{v}_j||^2}
\]
\[
= \frac{2(R^2 - \mathbf{v}_j \cdot \mathbf{o}_i - \mathbf{v}_i \cdot \mathbf{v}_j + \mathbf{v}_i \cdot \mathbf{o}_i)}{2R^2 - 2\mathbf{v}_i \cdot \mathbf{v}_j}
\]
\[
= \frac{R_i^2 - \mathbf{v}_i \cdot \mathbf{v}_j + \mathbf{v}_i \cdot \mathbf{o}_i}{R^2 - \mathbf{v}_i \cdot \mathbf{v}_j} \quad \text{(by (43))}
\]
Substituting into (44),

\[ w^i_j = \frac{R^2 - v_i \cdot v_j + v_i \cdot o_i}{R^2 - v_i \cdot v_j} \cdot v_i + \frac{R^2 - R^2 - v_i \cdot o_i}{R^2 - v_i \cdot v_j} \cdot v_j \]

\[ w^i_j \cdot v_i = \frac{R^2}{R^2 - v_i \cdot v_j} \cdot v_i + \frac{R^2}{R^2 - v_i \cdot v_j} \cdot v_j \]

\[ = \frac{R^2 R^2 + R^2 v_i \cdot o_i - R^2 v_i \cdot v_j - (v_i \cdot o_i) v_i \cdot v_j}{R^2 - v_i \cdot v_j} \]

\[ = \frac{(R^2 + v_i \cdot o_i)(R^2 - v_i \cdot v_j)}{R^2 - v_i \cdot v_j} \]

\[ = R^2 + v_i \cdot o_i \] (45)

Likewise, we have \( w^i_k = w^i_l = R^2 + v_i \cdot o_i \) in (41), and then (40) is established.

Finally, since the vectors \( w^i_j - w^i_k \parallel w^i_j - w^i_l \) as \( F_k \parallel F_l \), by (40), we can conclude that \( v_i \perp U_i \), and hence \( T_i \parallel U_i \).

Actually, looking back at (45), we can gain more insight about the situation:

\[ w^i_j \cdot v_i = R^2 + v_i \cdot o_i \]

\[ (w^i_j - o_i) \cdot v_i = R^2 \]

\[ = (w^i_j - o_i) \cdot (w^i_j - o_i) \]

\[ (w^i_j - o_i) \cdot (v_i + o_i - w^i_j) = 0 \]

which means that the vector from \( w^i_j \) to \( v_i + o_i \) is perpendicular to \( w^i_j - o_i \). This simple geometric interpretation of (45) motivates us to seek an elementary synthetic proof:

Proof. (Synthetic approach, rectilinear geometry) Refer to Figures 33, 34, 35, 36, 37 and 38.
Figure 33. Lemma 25
In Figure 33, $\alpha_{i,j}$ denotes the plane passing through $V_i, V_j$ and $O$, while $\beta_i$ denotes the plane containing $F_i$. The dashed segments lie on $\alpha_{i,j}$, while the solid segments lie off $\alpha_{i,j}$.

In Figure 34, $V_i$ is translated by the vector $o_i$ to $V_i'$, forming a parallelogram $OV_iV_i'O_i$, and

$$V_i'O_i \parallel \alpha_{i,j}, V_i'O_i \parallel V_i'O_i \quad \text{and} \quad V_i'O_i = V_i'O_i \quad (46)$$

In Figure 35, $V_i'O_i$ is projected onto $\alpha_{i,j}$ to $V_i''O_i'$, forming a rectangle $O_iV_iV_i'O_i'$, and

$$O_i'O_i' \perp \alpha_{i,j}, V_i''V_i'' \perp \alpha_{i,j}, O_i'O_i' = V_i''V_i', V_i'O_i||V_i''O_i' \quad \text{and} \quad V_i'O_i = V_i''O_i' \quad (47)$$

Also note that $\Delta V_jO_i'O_i' \cong \Delta W_j^iO_i'O_i'$, both $\perp_{\alpha_{i,j}}$, as $V_jO_i = W_j^iO_i$. As a result,

$$V_jO_i' = W_j^iO_i' \quad (48)$$

Moreover, we can prove that

$$\Delta V_jO_i'O_i' \cong \Delta O_i'W_j^iV_i'' \quad (49)$$

(the shaded triangles), as the next paragraph will explain. Figure 36 shows $\alpha_{i,j}$.

First of all, $V_jO_i' = W_j^iO_i'$ (marked by ‘=’) from (48), so that

$$\angle W_j^iV_jO_i' = \angle V_jW_j^iO_i' \quad \text{(marked by } \angle \text{ with } ' = ') \quad (50)$$
Next,
\[ V_j O = V_i O \]
\[ = V_i' O_i \quad \text{(by (46))} \]
\[ = V_i'' O_i' \quad \text{(by (47))} \]

and from (51) again, we have
\[ \angle V_i V_j O = \angle V_j V_i O \quad \text{(marked by } \angle \text{ with } ' - ') \]

Since \( V_i O \parallel V_i' O_i \parallel V_i'' O_i' \) by (46) and (47), we have
\[ \angle W_j^i O_i' V_i'' = \angle W_j^i \chi V_i, \text{ where } \chi \text{ is the intersection of } W_j^i O_i' \text{ and } V_i O \]
\[ = \angle V_i W_j^i O_i' - \angle V_j V_i O \]
\[ = \angle W_j^i V_j O_i' - \angle V_i V_j O \quad \text{(by (??) and (53))} \]
\[ = \angle O V_j O_i' \]

Hence, (49) follows from (48), (52) and (54).

Back to Figure 35. From (49), we have \( O O_i' = W_j^i V_i'' \). Plus \( O_i O_i' \perp \alpha_{i,j} \) and \( V_i' V_i'' \perp \alpha_{i,j} \) from (47), we can then prove that
\[ \Delta V_j O_i O \cong \Delta O_i W_j^i V_i' \]

Note that as \( O O_i \perp V_j \), (55) will imply that \( W_j^i O_i \perp W_j^i V_i' \) — the simple geometric interpretation of (45) which has motivated the present proof. But now we have proved something stronger. Similar to (55), we will also have
\[ \Delta V_k O_i O \cong \Delta O_i W_k^i V_i' \text{ and } \Delta V_i O_i O \cong \Delta O_i W_i^i V_i' \]

(56)
In Figure 37, the three shaded triangles $\Delta V_j O_i O, \Delta V_k O_i O$ and $\Delta V_l O_i O$ are congruent as $O O_i \perp F_i$ and $O$ is equidistant to $V_j, V_k, V_l$. Therefore, by (55) and (56),

$$\Delta O_i W_j^i V_i' \cong \Delta O_i W_k^i V_i' \cong \Delta O_i W_l^i V_i'$$

(57)
as shown as the shaded triangles in Figure 38. Because of (57), $W_j^i, W_k^i$ and $W_l^i$ will project onto the line $V_i'O_i$ to the same point $P$ as shown. This shows that the orthic plane $U_i$, i.e. the plane containing $W_j^i, W_k^i$ and $W_l^i$, intersects the line $V_i'O_i$ perpendicularly at $P$.

We have just proved more than what Lemma 25 has claimed. However, the final conclusion that $U_i$ intersects the line $V_i'O_i$ perpendicularly at $P$ will play a crucial role when proving Theorem 28. If only the parallelism between $T_i$ and $U_i$ is concerned, Lemma 25 actually admits a more elegant synthetic proof, through some slightly higher geometry:

**Proof.** (Synthetic approach, circle geometry) Refer to Figure 39, 40, 41, 42 and 43.

In Figure 39, $\pi_i$ is the plane containing the face $F_i$. Its intersection with the sphere $S_i$ (centred at $O_i$ with radius $V_j O_i$) is the circle denoted by $C_{i,1}$. Then, $V_j, V_k, W_j^i$ and $W_k^i$ are concyclic on $C_{i,1}$, so that $\alpha = \beta$. 
In Figure 40, $C_l$ denotes the circle of intersection of the plane $\pi_l$ and the circumsphere $S^{ci}$ of $\Delta$. The tangent line to $C_l$ at $V_i$ on $\pi_l$ is denoted by $l_{l,i}$. By the tangency, $\beta = \gamma$. As a result, $\alpha = \gamma$.

In Figure 41, recall from Proposition 4 that $OO_l \perp \pi_l$. But as $l_{l,i}$ is tangent to $C_l$ on $\pi_l$, we also have $l_{l,i} \perp V_iO_l$. As a result, $l_{l,i} \perp V_iO$.

In Figure 42, since $l_{l,i} \perp V_iO$ as we have just proved, $l_{l,i}$ is indeed tangent to the circumsphere $S^{ci}$ of $\Delta$, so $l_{l,i}$ actually resides on the tangential plane $T_i$ of $\Delta$. Then by $\alpha = \gamma$, we have $l_{l,i} \parallel W_j^iW_k^i$, which is a pair of parallel lines lying on $T_i$ and $U_i$ respectively.
In Figure 43, we run the same argument for the plane $\pi_j$ containing the face $F_j$ to get $l_{j,i} \parallel W_k^i W_l^i$, which is another pair of parallel lines lying on $T_i$ and $U_i$ respectively, and we can conclude that $T_i \parallel U_i$. \hfill \Box

![Figure 43. Lemma 25](image)

As a consequence of Lemma 25, we can define:

**Definition 26.** (Orthic tetrahedron) The orthic tetrahedron $\Delta_{ot}$ of a tetrahedron $\Delta$ is the tetrahedron enclosed by its four orthic planes. Similar to tangential tetrahedron in Definition 23, it is well-defined only if $\Delta$ is acute-angled or obtuse-angled, and is undefined if $\Delta$ is right-angled. See Figures 44 (acute-angled case) and 45 (obtuse-angled case) for illustrations.

By Definition 26, Lemma 25 and Theorem 20, $\Delta_{tg}$ and $\Delta_{ot}$ are homothetic, and we define the following:
Definition 27. \((\chi_{25} \text{ and orthic inexcercntre})\) The point \(\chi_{25}\) of a tetrahedron \(\Delta\) is defined as the homothetic centre \(Z(\Delta_{tg}, \Delta_{ot})\) between its tangential tetrahedron \(\Delta_{tg}\) and its orthic tetrahedron \(\Delta_{ot}\). This definition is completely analogous to that of \(\chi_{25}\) of triangles [15].

The orthic inexcercntre \(K\) of \(\Delta\) is defined as

1. the incentre \(T(\Delta_{ot})\) of \(\Delta_{ot}\) if \(\Delta\) is acute-angled,
2. the excentre \(T_i(\Delta_{ot})\) of \(\Delta_{ot}\) if \(\Delta\) is obtuse-angled, where \(V_i\) is the obtuse vertex of \(\Delta\).

See Figure 46 (acute-angled case) and 47 (obtuse-angled case) for illustrations.
We now prove the important properties, one algebraic and one geometric, of ortho-
centre of triangles that are carried over through generalizing as orthic in-
excentre of tetrahedra.

**Theorem 28.** (Collinearity of orthic in-excentre, circumcentre and \(\chi_{25}\)) The orthic
in-excentre \(K\), the circumcentre \(O\) and \(\chi_{25}\) of a tetrahedron are collinear.

**Proof.** It only suffices to show that \(K\) and \(O\) of a tetrahedron \(\Delta\) are corresponding
points under the homothety between \(\Delta_{tg}\) and \(\Delta_{ot}\). Recalling from Definition 23
that

\[
O(\Delta) = I(\Delta_{tg}) \quad \text{or} \quad O(\Delta) = I_i(\Delta_{tg})
\]

respectively in the acute-angled or obtuse-angled cases, as well as from Definition 27
that

\[
K(\Delta) = I(\Delta_{ot}) \quad \text{or} \quad K(\Delta) = I_i(\Delta_{ot})
\]

accordingly, we may indeed try to show that incentre and excentre are preserved
under homothetic transformations.

In fact, if \(i\) satisfies (9) to become the incentre \(I(\Delta(W))\) of a tetrahedron \(\Delta(W) :=
[W_0, W_1, W_2, W_3]\), where \(n_0, n_1, n_2, n_3\) are the inward normal vectors of \(\Delta(W)\),
then under any homothetic transformation \(T\) of the form (27) or (28), by Theorem 20,
the inward normal vectors of \(T(\Delta(W)) = [T(W_0), T(W_1), T(W_2), T(W_3)]\) will also
be \(n_0, n_1, n_2, n_3\), and

\[
n_i \cdot (T(i) - T(p_i)) = n_i \cdot (ti - tp_i) = tn_i \cdot (i - p_i)
\]

As a result, \(T(i)\) will satisfy

\[
n_0 \cdot (T(i) - T(p_0)) = n_1 \cdot (T(i) - T(p_1)) = n_2 \cdot (T(i) - T(p_2)) = n_3 \cdot (T(i) - T(p_3))
\]

to become the incentre \(I(T(\Delta(W)))\) of \(T(\Delta(W))\). Therefore, incentre is preserved
under homothetic transformations. Similarly, letting \(\{i, j, k, l\} = \{0, 1, 2, 3\}\), if \(ii\)
satisfies (13) to become the excentre \(I_i(\Delta(W))\) of \(\Delta(W)\), then \(T(i)\) will satisfy

\[
n_i' \cdot (T(i) - T(p_0)) = n_j \cdot (T(i) - T(p_j)) = n_k \cdot (T(i) - T(p_k)) = n_l \cdot (T(i) - T(p_l))
\]

to become the excentre \(I_i(T(\Delta(W)))\) of \(T(\Delta(W))\). Therefore, excentre is preserved
under homothetic transformations.

Hence, incentre and excentre are preserved under homothetic transformations, and
\(K, O\) and \(\chi_{25}\) are collinear.

We have proved the geometric property of \(K\), now we turn to prove the algebraic
property of \(K\).

**Theorem 29.** (Vector representation of orthic in-excentre) Let \(\Delta = [V_0, V_1, V_2, V_3]\)
be a tetrahedron. If its circumcentre \(O\) is the origin, then its orthic in-excentre \(K\)
of \(\Delta\) can be expressed as

\[
k = o_0 + o_1 + o_2 + o_3
\]

where \(O_t\) denotes the circumcentre of the face \(F_i\).
Proof. We shall compute the equations of the orthic planes $U_0, U_1, U_2, U_3$, and then verify that the point represented by the right-hand side of (58) satisfies the incentre or excentre requirement for $K$ in Definition 27.

Let $\{i, j, k, l\} = \{0, 1, 2, 3\}$. Recall from (39) that the equation of $T_i$ is given by $(-v_i/R) \cdot x = -R$. Note that we have multiplied $-1/||v_i|| = -1/R$ to both sides, which will make the normal vector $-v_i/R$ an inward one if $O$ and $V_i$ lie on the same side of $F_i$, and an outward one if $O$ and $V_i$ lie on opposite sides of $F_i$. Therefore, $-v_i/R$ will be the required normal vector of $U_i$ for both cases in Definition 27.

To compute the equation of $U_i$, we may compute the position vector $p$ of $P$ in Figure 38. By (55) in the proof of Lemma 25 and referring to Figure 35, if $Q$ is the projection of $O_i$ onto $V_j O$, then we have

$$O_i P = V_j Q = \frac{(o_i - v_j) \cdot (-v_j)}{||v_j||} = \frac{R^2 - v_j \cdot o_i}{R} = \frac{R_i^2}{R}$$

with the help of (43), and then

$$p = o_i + \frac{R_i^2}{R} \frac{v_i}{||v_i||} = o_i + \frac{R_i^2}{R^2} v_i$$

as shown in Figure 34. Therefore, the required equation of $U_i$ is

$$-\frac{v_i}{R} \cdot \left( x - o_i - \frac{R_i^2}{R^2} v_i \right) = 0$$

By substituting

$$x = o_0 + o_1 + o_2 + o_3 \quad (59)$$

into the left-hand side, its distance from

$$U_i = -\frac{v_i}{R} \cdot \left( o_j + o_k + o_l - \frac{R_i^2}{R^2} v_i \right)$$

$$= -\frac{v_i \cdot o_j - v_i \cdot o_k - v_i \cdot o_l}{R} + \frac{R_i^2}{R}$$

$$= \frac{R_i^2 + R_j^2 + R_k^2 + R_l^2 - 3R^2}{R} \quad (by \; (43)) \quad (60)$$

Note that (60) is the same for $U_i, U_j, U_k, U_l$, so (59) is equidistant from all $U_i, U_j, U_k, U_l$, and (58) is verified.

Note that (58) is well-defined even for right-angled tetrahedra, thus Theorem 29 enables us define orthic incentre by (58) for any tetrahedron. Also, if the circumcentre $O$ is not assumed to be the origin, then (58) should be modified as

$$k - o = (o_0 - o) + (o_1 - o) + (o_2 - o) + (o_3 - o)$$

$$k = o_0 + o_1 + o_2 + o_3 - 3o \quad (61)$$
Corollary 30. (Vector representation of $\chi_{25}$) Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron. If its circumcentre $O$ is the origin, then its $\chi_{25}$ of $\Delta$ can be expressed as

$$\chi_{25} = \frac{R^2}{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 2R^2 k},$$

(62)

where $O_i$ and $R_i$ denote the circumcentre and the circumradius of the face $F_i$ respectively.

Proof. While the common distance from $O(\Delta)$ to the faces $T_0, T_1, T_2, T_3$ of $\Delta_{tg}$ is $R$, that from $K(\Delta)$ to the faces $U_0, U_1, U_2, U_3$ of $\Delta_{ot}$ is as in (60). Therefore,

$$\chi_{25} = \frac{R}{R + \frac{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 3R^2}{R} k},$$

and we obtain (62).

Note that if the circumcentre $O$ is not assumed to be the origin, then (62) should be modified as

$$\chi_{25} - o = \frac{R^2}{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 2R^2} (k - o)$$

$$\chi_{25} = \frac{R^2}{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 2R^2} k + \frac{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 3R^2}{R^2} o$$

(63)

5. Tetrahedron Centres and Barycentric Coordinates

We have been talking about tetrahedron centres, but unlike triangle centres which have already been clearly defined in [14], apparently no precise definition of tetrahedron centre has ever been written down yet. In [8], the author did define and use the terminology when formulating and investigating the so-called centre conjecture, which has a completely different purpose.

Here we will express our own perception of this seemingly immediate generalization of triangle centres to tetrahedra, and will formulate a preliminary framework of tetrahedron centres. Moreover, the notion of (homogeneous) barycentric coordinates have provided a powerful tool for analyzing triangle centres problems, as shown in [22]. This inspired us to introduce barycentric function to construct tetrahedron centres.

In Section 5.1, we will lay down precise definitions of triangle centres and tetrahedron centres in the space from our perspective in Definition ???. Then, we will prove Lemma 33 about behaviours of tetrahedra under similarity transformations, which will also justify that Definition ?? is well-defined. Proposition 35 will verify that all the aforementioned tetrahedron centres satisfy Definition ???. Propositions 36 and 36 will provide simple ways to generate tetrahedron centres from others.
In Section 5.2, a general analytic form for tetrahedron centres will be constructed through the concept of barycentric function which will be introduced in Definition 37. Proposition 38 and Theorem ?? will prove that such construction completely characterizes all tetrahedron centres.

5.1. Defining Tetrahedron Centres as Functions

In [14], functions with certain homogeneity and symmetry were used to define triangle centres in the plane. Inspired by that approach, here we express our own perception of triangles centres and tetrahedra centres in the space, which could be described in a more succinct manner.

**Definition 31.** (Similarity transformation) A similarity transformation $T$ of $\mathbb{R}^3$ is a function $T: \mathbb{R}^3 \rightarrow \mathbb{R}^3$ of the form

$$T(u) = tAu + b \quad \text{for } u \in \mathbb{R}^3$$

(64)

where $t \in \mathbb{R}\setminus\{0\}$, $A$ is a $3 \times 3$ orthogonal matrix (i.e. $A^T A = I$ or $A^{-1} = A^T$) and $b$ is a vector in $\mathbb{R}^3$.

Note that similarity transformations $T$ of the form (64) have the very useful properties

$$\begin{align*}
(T(u) - T(v)) \cdot (T(u') - T(v')) &= tA(u - v) \cdot tA(u' - v') \\
&= t^2(A(u - v))^T(A(u' - v')) \\
&= t^2(u - v)^T A^T A(u' - v') \\
&= t^2(u - v)^T (u' - v') \\
&= t^2(u - v) \cdot (u' - v')
\end{align*}$$

(65)

and

$$\begin{align*}
Aw \cdot (T(u) - T(v)) &= Aw \cdot tA(u - v) \\
&= t(Aw)^T(A(u - v)) \\
&= tw^T A^T A(u - v) \\
&= tw^T (u - v) \\
&= tw \cdot (u - v)
\end{align*}$$

(66)

**Definition 32.** (Triangle centre and tetrahedron centre) Let $S^2$ denote the set of all triangles in $\mathbb{R}^3$. Then, a triangle centre $\chi^2/\chi^2$ is a function $\chi^2 : S^2 \rightarrow \mathbb{R}^3$ assigning to each $\Delta^2 \in S^2$ a point in the plane containing $\Delta^2$, such that it is equivariant under similarity transformations $T$ of $\mathbb{R}^3$, i.e.

$$T(\chi^2(\Delta^2)) = \chi^2(T(\Delta^2)) \quad \text{for } \Delta^2 \in S^2$$

(67)
Let $S^3$ denote the set of all tetrahedra in $\mathbb{R}^3$. Then, a tetrahedron centre $\chi^3 / x^3$ is a function $\chi^3 : S^3 \to \mathbb{R}^3$ such that it is equivariant under similarity transformations $T$ of $\mathbb{R}^3$, i.e.

$$T(x^3(\Delta^3)) = x^3(T(\Delta^3)) \quad \text{for } \Delta^2 \in S^3$$

(68)

Let $S = S^2 \cup S^3$ denote the set of all triangles and tetrahedra in $\mathbb{R}^3$. Then, joining the above triangle and tetrahedron centres by

$$\chi|_{S^2} = \chi^2 \quad \text{and} \quad \chi|_{S^3} = \chi^3$$

defines a centre $\chi / x$ as a function $\chi : S \to \mathbb{R}^3$.

The requirement that $\chi^2(\Delta^2)$ lies in the plane containing $\Delta^2$ can, in particular, avoid the circumcentre of a triangle from becoming an arbitrary point in a line when it is described as “a point equidistant from the vertices of the triangle”. For some non-classical centres such as the Fermat-Torricelli point which is defined only for a subclass of triangles, the domains may need modifications.

The equivariances (67) and (68) require that for any triangle or tetrahedron and any similarity transformation of the space, the centre of the transformed triangle or tetrahedron is exactly the transformed centre of the original triangle or tetrahedron. Moreover, the right-hand sides of these two equations require that $T(\Delta^2)$ and $T(\Delta^3)$ are also a triangle and a tetrahedron respectively – they are, as we now prove:

**Lemma 33.** (Triangle and tetrahedron under similarity transformation) Let $\Delta = [V_0, V_1, V_2, V_3]$ be a tetrahedron and $T$ be a similarity transformation of $\mathbb{R}^3$. Then,

(a) $\{T(v_0), T(v_1), T(v_2), T(v_3)\}$ are affinely independent,
(b) $T(\Delta) = [T(V_0), T(V_1), T(V_2), T(V_3)]$,
(c) $T(E_{i,j}) = [T(V_i), T(V_j)]$, where $i \neq j$, and
(d) $T(F_i) = [T(V_j), T(V_k), T(V_l)]$, where $\{i, j, k, l\} = \{0, 1, 2, 3\}$.

**Proof.** Under similarity transformation $T$ of the form (64), we have

$$T(v_i) - T(v_0) = tA(v_i - v_0) \quad \text{for } i = 1, 2, 3$$

Suppose there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}$ such that

$$\alpha_1(T(v_1) - T(v_0)) + \alpha_2(T(v_2) - T(v_0)) + \alpha_3(T(v_3) - T(v_0)) = 0$$

$$tA(\alpha_1(v_1 - v_0) + \alpha_2(v_2 - v_0) + \alpha_3(v_3 - v_0)) = 0$$

$$\alpha_1(v_1 - v_0) + \alpha_2(v_2 - v_0) + \alpha_3(v_3 - v_0) = A^{-1}0 = 0$$

($A$ is invertible)

Since $\{v_1 - v_0, v_2 - v_0, v_3 - v_0\}$ are linearly independent, we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$. Therefore, the transformed vertices $\{T(v_0), T(v_1), T(v_2), T(v_3)\}$ are affinely independent too, hence proving (a).
As mentioned in Section 1.3, any point \( u \) in the tetrahedron can be represented as a convex combination as in (1), so we have

\[
T(u) = tA(\lambda_0v_0 + \lambda_1v_1 + \lambda_2v_2 + \lambda_3v_3) + b
= \lambda_0(tAv_0 + b) + \lambda_1(tAv_1 + b) + \lambda_2(tAv_2 + b) + \lambda_3(tAv_3 + b)
= \lambda_0T(v_0) + \lambda_1T(v_1) + \lambda_2T(v_2) + \lambda_3T(v_3)
\]

This shows that as \( \lambda_0, \lambda_1, \lambda_2, \lambda_3 \) run through the condition in (1) so that \( u \) runs through \( \Delta \), \( T(u) \) will run through every point the tetrahedron \([T(v_0), T(v_1), T(v_2), T(v_3)]\) in a one-to-one correspondence manner. More precisely, similarity transformations preserve convex combination. Hence, (b), (c) and (d) follow.

We now verify that centroid, circumcentre, incentre, excentre, Monge point, quasi-orthocentre, antimedial circumcentre, orthic inexcenTre and \( \chi_{25} \) are tetrahedron centres:

**Proposition 34.** (A source of tetrahedron centres) Centroid, circumcentre, incentre, Monge point, quasi-orthocentre, antimedial circumcentre, orthic inexcenTre and \( \chi_{25} \) are tetrahedron centres.

Excentres form ‘a group of’ tetrahedron centres. Here, ‘a group of’ tetrahedron centres is defined as a function \( \chi : S^3 \to P_4(\mathbb{R}^3) \), where \( P_4(\mathbb{R}^3) \) denotes the set of all the 4-element subsets of \( \mathbb{R}^3 \), such that it is equivariant under similarity transformations of \( \mathbb{R}^3 \).

**Proof.** That is to verify that \( G, O, I, \{I_0, I_1, I_2, I_3\}, M, Q, J, K \) are equivariant under similarity transformations of \( \mathbb{R}^3 \). To this end, let \( \Delta = [V_0, V_1, V_2, V_3] \) be a tetrahedron, and let \( T \) be any similarity transformation of the form (64). Recall from Lemma 33 that \( T(\Delta) = [T(V_0), T(V_1), T(V_2), T(V_3)], T(E_{i,j}) = [T(V_i), T(V_j)] \) for \( i \neq j \), and \( T(F_i) = [T(V_j), T(V_k), T(V_l)] \) for \( \{i, j, k, l\} = \{0, 1, 2, 3\} \). We shall show that

\[
T(x(\Delta)) = x(T(\Delta)) \quad \text{for} \quad G, O, I, \{I_0, I_1, I_2, I_3\}, M, Q, J, K
\]

**Centroid:**

\[
T(g(\Delta)) = tA \left( \frac{1}{4}(v_0 + v_1 + v_2 + v_3) \right) + b \quad \text{(by (2))}
= \frac{1}{4}(tAv_0 + b) + \frac{1}{4}(tAv_1 + b) + \frac{1}{4}(tAv_2 + b) + \frac{1}{4}(tAv_3 + b)
= \frac{1}{4}T(v_0) + \frac{1}{4}T(v_1) + \frac{1}{4}T(v_2) + \frac{1}{4}T(v_3) \quad \text{(by (2))}
= g(T(\Delta))
\]
**Circumcentre:** Recall from (3) that
\[ ||\mathbf{o}(\Delta) - \mathbf{v}_0|| = ||\mathbf{o}(\Delta) - \mathbf{v}_i|| = ||\mathbf{o}(\Delta) - \mathbf{v}_2|| = ||\mathbf{o}(\Delta) - \mathbf{v}_3|| \]
But by (65),
\[ ||T(\mathbf{o}(\Delta)) - T(\mathbf{v}_i)||^2 = t^2||\mathbf{o}(\Delta) - \mathbf{v}_i||^2 \]
so
\[ ||T(\mathbf{o}(\Delta)) - T(\mathbf{v}_0)|| = ||T(\mathbf{o}(\Delta)) - T(\mathbf{v}_1)|| \]
\[ = ||T(\mathbf{o}(\Delta)) - T(\mathbf{v}_2)|| \]
\[ = ||T(\mathbf{o}(\Delta)) - T(\mathbf{v}_3)|| \]
\[ = tR \]  \hspace{1cm} (70)
where \( R \) is the circumradius of \( \Delta \). Therefore,
\[ T(\mathbf{o}(\Delta)) = \mathbf{o}(T(\Delta)) \]  \hspace{1cm} (71)

**Incentre:** Recall from (9) that
\[ \mathbf{n}_0 \cdot (\mathbf{i}(\Delta) - \mathbf{p}_0) = \mathbf{n}_1 \cdot (\mathbf{i}(\Delta) - \mathbf{p}_1) = \mathbf{n}_2 \cdot (\mathbf{i}(\Delta) - \mathbf{p}_2) = \mathbf{n}_3 \cdot (\mathbf{i}(\Delta) - \mathbf{p}_3), \]
where \( \mathbf{n}_i \) is the inward normal vector of the face \( F_i \) and \( \mathbf{p}_i \) is a point \( F_i \). Then,
\[ \mathbf{\tilde{n}}_i := \text{sgn}(t)A\mathbf{n}_i \]
will become the inward normal vector of the face \( T(F_i) \). It is because \[ ||\mathbf{\tilde{n}}_i||^2 = ||\text{sgn}(t)A\mathbf{n}_i||^2 = (A\mathbf{n}_i)^T(A\mathbf{n}_i) = \mathbf{n}_i^T(A^T A)\mathbf{n}_i = \mathbf{n}_i^T\mathbf{n}_i = ||\mathbf{n}_i||^2 = 1, \]
and by (66),
\[ \mathbf{\tilde{n}}_i \cdot (T(\mathbf{v}_j) - T(\mathbf{v}_k)) = \text{sgn}(t)t \mathbf{n}_i \cdot (\mathbf{v}_j - \mathbf{v}_k) = |t|\mathbf{n}_i \cdot \mathbf{e}_{j,k}, \]
so that \[ \mathbf{\tilde{n}}_i \cdot (T(\mathbf{v}_j) - T(\mathbf{v}_k)) = 0 \] for \( j,k \neq i \) and \[ \mathbf{\tilde{n}}_i \cdot (T(\mathbf{v}_i) - T(\mathbf{v}_j)) > 0 \] for \( j \neq i \). By (66) again, we will also have
\[ \mathbf{\tilde{n}}_i \cdot (T(\mathbf{i}(\Delta)) - T(\mathbf{p}_i)) = |t|\mathbf{n}_i \cdot (\mathbf{i}(\Delta) - \mathbf{p}_i), \]
so
\[ \mathbf{\tilde{n}}_0 \cdot (T(\mathbf{i}(\Delta)) - T(\mathbf{p}_0))) = \mathbf{\tilde{n}}_1 \cdot (T(\mathbf{i}(\Delta)) - T(\mathbf{p}_1)) \]
\[ = \mathbf{\tilde{n}}_2 \cdot (T(\mathbf{i}(\Delta)) - T(\mathbf{p}_2)) \]
\[ = \mathbf{\tilde{n}}_3 \cdot (T(\mathbf{i}(\Delta)) - T(\mathbf{p}_3)) \]
Therefore,
\[ T(\mathbf{i}(\Delta)) = \mathbf{i}(T(\Delta)) \]

**Excentre:** Let \( \{i,j,k,l\} = \{0,1,2,3\} \). Recall from (13) that
\[ \mathbf{n}_i' \cdot (\mathbf{i}_i(\Delta) - \mathbf{p}_i) = \mathbf{n}_j \cdot (\mathbf{i}_i(\Delta) - \mathbf{p}_j) = \mathbf{n}_k \cdot (\mathbf{i}_i(\Delta) - \mathbf{p}_k) = \mathbf{n}_l \cdot (\mathbf{i}_i(\Delta) - \mathbf{p}_l) \]
Defining \( \mathbf{\tilde{n}}_i' = -\mathbf{n}_i' \), by (66),
\[ \mathbf{\tilde{n}}_i' \cdot (T(\mathbf{i}_i(\Delta)) - T(\mathbf{p}_i)) = |t|\mathbf{n}_i' \cdot (\mathbf{i}_i(\Delta) - \mathbf{p}_i) \quad \text{and} \]
\[ \mathbf{\tilde{n}}_j' \cdot (T(\mathbf{i}_i(\Delta)) - T(\mathbf{p}_j)) = |t|\mathbf{n}_j \cdot (\mathbf{i}_i(\Delta) - \mathbf{p}_j) \quad \text{for} \ j \neq i, \]
\[ \hat{n}'_i \cdot (T(i(\Delta)) - T(p_i)) = \hat{n}'_j \cdot (T(i(\Delta)) - T(p_j)) = \hat{n}'_k \cdot (T(i(\Delta)) - T(p_k)) = \hat{n}'_l \cdot (T(i(\Delta)) - T(p_l)) \]

However, we may not conclude that \( T(i_i(\Delta)) = i_i(T(\Delta)) \), but only
\[ T(i_i(\Delta)) = i_{\sigma(i)}(T(\Delta)), \]
where \( \sigma \) is a permutation of \( \{0, 1, 2, 3\} \), as \( T(\Delta) = [T(V_0), T(V_1), T(V_2), T(V_3)] = [T(V_{\sigma(0)}), T(V_{\sigma(1)}), T(V_{\sigma(2)}), T(V_{\sigma(3)})] \) in general. Nevertheless, we must have
\[ T(\{i_0(\Delta), i_1(\Delta), i_2(\Delta), i_3(\Delta)\}) = \{i_0(T(\Delta)), i_1(T(\Delta)), i_2(T(\Delta)), i_3(T(\Delta))\} \]

**Monge point:**
\[ T(m(\Delta)) = tA(2g(\Delta) - o(\Delta)) + b \quad (\text{by (19)}) \]
\[ = 2(tAg(\Delta) + b) - (tAo(\Delta) + b) \]
\[ = 2T(g(\Delta)) - T(o(\Delta)) \]
\[ = 2g(T(\Delta)) - o(T(\Delta)) \quad (\text{by (69) and (71)}) \]
\[ = m(T(\Delta)) \quad (\text{by (19)}) \]

**k-quasi-orthocentre:**
\[ T(q_k(\Delta)) = tA \left( \frac{4}{3 + k} g(\Delta) - \frac{1 - k}{3 + k} o(\Delta) \right) + b \quad (\text{by (26)}) \]
\[ = \frac{4}{3 + k} (tAg(\Delta) + b) - \frac{1 - k}{3 + k} (tAo(\Delta) + b) \]
\[ = \frac{4}{3 + k} (tAg(\Delta) + b) - \frac{1 - k}{3 + k} T(o(\Delta)) \]
\[ = \frac{4}{3 + k} (tAg(\Delta) + b) - \frac{1 - k}{3 + k} o(T(\Delta)) \quad (\text{by (69) and (71)}) \]
\[ = q_k(T(\Delta)) \quad (\text{by (26)}) \]

**Antimedial circumcentre:**
\[ T(j(\Delta)) = tA(4g(\Delta) - 3o(\Delta)) + b \quad (\text{by (38)}) \]
\[ = 4(tAg(\Delta) + b) - 3(tAo(\Delta) + b) \]
\[ = 4T(g(\Delta)) - 3T(o(\Delta)) \]
\[ = 4g(T(\Delta)) - 3o(T(\Delta)) \quad (\text{by (69) and (71)}) \]
\[ = j(T(\Delta)) \quad (\text{by (38)}) \]

**Orthic inexceneter:** Similar to the proof about circumcentre above, we have
\[ T(o(F_i)) = o(T(F_i)), \quad (72) \]
where this $\circ$ refers to circumcentre of triangles, and the circumradius of the face $T(F_i)$ of $T(\Delta)$

$$R_i(T(\Delta)) = tR_i$$

where $R_i$ is the circumradius of $F_i$. Then,

$$T(k(\Delta)) = tA(o_0(\Delta) + o_1(\Delta) + o_2(\Delta) + o_3(\Delta) - 3o(\Delta)) + b \quad \text{(by (61))}$$

$$= tA(o(F_0) + o(F_1) + o(F_2) + o(F_3) - 3o(\Delta)) + b$$

$$= (tA\circ(F_0) + b) + (tA\circ(F_1) + b) + (tA\circ(F_2) + b) + (tA\circ(F_3) + b)$$

$$3(tA\circ(\Delta) + b)$$

$$= T(o(F_0)) + T(o(F_1)) + T(o(F_2)) + T(o(F_3)) - 3T(o(\Delta))$$

$$= o(T(F_0)) + o(T(F_1)) + o(T(F_2)) + o(T(F_3)) - 3o(T(\Delta))$$

(by (71) and (72))

$$= o_0(T(\Delta)) + o_1(T(\Delta)) + o_2(T(\Delta)) + o_3(T(\Delta)) - 3o(T(\Delta))$$

$$= k(T(\Delta)) \quad \text{(by (61))}$$

$\chi_{25}$:

$$T(x_{25}(\Delta))$$

$$= tA\left(\frac{R^2}{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 2R^2}k(\Delta)\right.$$\n
$$\left.\frac{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 3R^2}{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 2R^2}o(\Delta)\right) + b \quad \text{(by (63))}$$

$$= \frac{R^2}{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 2R^2} (tAk(\Delta) + b) +$$

$$\frac{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 3R^2}{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 2R^2} (tA\circ(\Delta) + b)$$

$$= \frac{R^2}{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 2R^2} T(k(\Delta)) + \frac{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 3R^2}{R_0^2 + R_1^2 + R_2^2 + R_3^2 - 2R^2} T(o(\Delta))$$

$$= \frac{(tR_0)^2 + (tR_1)^2 + (tR_2)^2 + (tR_3)^2 - 2(tR)^2}{(tR)^2} k(T(\Delta))$$

$$+ \frac{(tR_0)^2 + (tR_1)^2 + (tR_2)^2 + (tR_3)^2 - 3(tR)^2}{(tR)^2} o(T(\Delta))$$

(by (71) and (74))

$$= \chi_{25}(T(\Delta)) \quad \text{(by (63), (70) and (73))}$$

Hence, centroid, circumcentre, incentre, excentre, Monge point, quasi-orthocentre, antimedial circumcentre, orthic in-excentre and $\chi_{25}$ are tetrahedron centres.

Proposition 34 motivates us to prove two general ways to construct tetrahedron centres from others:
Proposition 35. (Affine combination of tetrahedron centres) An affine combination of tetrahedron centres is also a tetrahedron centre.

Proof. Let $\chi/x_1, \ldots, \chi_m/x_m$ be some tetrahedron centres, and let $\gamma_1, \ldots, \gamma_m \in \mathbb{R}$ such that $\sum_{i=1}^m \gamma_i = 1$. Consider the affine combination $\chi/x$ defined by

$$x(\Delta) := \sum_{i=1}^m \gamma_i x_i(\Delta) \quad \text{for } \Delta \in S^3$$

Under a similarity transformation $T$ as described in (64),

$$T(x(\Delta)) = tA \sum_{i=1}^m \gamma_i x_i(\Delta) + b$$

$$= \sum_{i=1}^m \gamma_i tAx_i(\Delta) + \sum_{i=1}^m \gamma_i b$$

$$= \sum_{i=1}^m \gamma_i (tAx_i(\Delta) + b)$$

$$= \sum_{i=1}^m \gamma_i T(x_i(\Delta))$$

$$= \sum_{i=1}^m \gamma_i x_i(T(\Delta))$$

$$= x(T(\Delta)),$$

and hence $\chi$ is equivariant under similarity transformations. \hfill \Box

While Proposition 35 may be quite intuitive, the next will be less obvious.

Proposition 36. (Combination of facial centres) If $\chi/x$ is a triangle centre and $\mathcal{Y}/y$ is a tetrahedron centre, then $Z/z$ defined as

$$z(\Delta) := y(\Delta) + \sum_i (x(F_i) - y(\Delta)) \quad \text{for } \Delta \in S^3$$

is also a tetrahedron centre.
Proof. Under a similarity transformation $T$ as described in (64),

$$T(z(\Delta)) = tA \left( y(\Delta) + \sum_i (x(F_i) - y(\Delta)) \right) + b$$

$$= tAy(\Delta) + b + \sum_i (tAx(F_i) - tAy(\Delta))$$

$$= T(y(\Delta)) + \sum_i (T(x(F_i)) - T(y(\Delta)))$$

$$= y(T(\Delta)) + \sum_i (T(F_i) - T(\Delta))$$

$$= z(T(\Delta)),$$

and hence $Z$ is equivariant under similarity transformations. In the last step, Lemma 33(d) is used so that $T(\Delta) = [T(F_0), T(F_1), T(F_2), T(F_3)]$.

Figure 48 illustrates Proposition 36: the point $Z(\Delta)$ is obtained by translating the point $Y(\Delta)$ by the resultant vector (solid arrow) of the other four vectors (dotted arrows).

Figure 48. Proposition 36

5.2. Constructing Tetrahedron Centres Using Barycentric Functions

Each point in the plane containing a triangle in the space can be expressed uniquely in barycentric coordinates with respect to the vertices of the triangle, so it is desirable to generalize this idea to the space and subsequently express the tetrahedron centres in barycentric coordinates with respect to the vertices of the tetrahedron.

Definition 37. (Barycentric function) Let $\mathbb{V}$ be the set of all affinely independent quadruples of vectors in $\mathbb{R}^3$. Then, $\lambda : \mathbb{V} \to \mathbb{R}$ is a barycentric function if:
(i) Invariance under similarity transformations:
\[ \lambda(T(x_0), T(x_1), T(x_2), T(x_3)) = \lambda(x_0, x_1, x_2, x_3) \]
for \((x_0, x_1, x_2, x_3) \in V\), for any similarity transformation \(T\) of \(\mathbb{R}^3\).

(ii) Symmetry in the second, third and fourth variables:
\[ \lambda(x_0, x_1, x_2, x_3) = \lambda(x_0, x_1, x_2, x_3) \]
for \((x_0, x_1, x_2, x_3) \in V\), where \(\{i, j, k\} = \{1, 2, 3\}\).

(iii) Normalization:
\[ \lambda(x_0, x_1, x_2, x_3) + \lambda(x_1, x_2, x_3, x_0) + \lambda(x_2, x_3, x_0, x_1) + \lambda(x_3, x_0, x_1, x_2) \]
\[ = 1 \]
for \((x_0, x_1, x_2, x_3) \in V\).

Because of the symmetry (ii), we can abbreviate \(\lambda(x_0, x_1, x_2, x_3)\) as \(\lambda(x_0)\), so that (iii) can be rewritten as
\[ \lambda(x_0) + \lambda(x_1) + \lambda(x_2) + \lambda(x_3) = 1 \]
when the context is clear. It is then routine to check that barycentric functions can generate tetrahedron centres:

**Proposition 38.** (Barycentric functions generate tetrahedron centres) If \(\lambda\) is a barycentric function, then \(\chi/\mathbf{x}\) defined as
\[ \mathbf{x}(\Delta) := \sum_i \lambda(v_i)v_i \quad \text{for} \quad \Delta = [V_0, V_1, V_2, V_3] \in \mathbb{S}^3 \]
is a tetrahedron centre.

**Proof.** Under a similarity transformation \(T\) as described in (64),
\[ T(\mathbf{x}(\Delta)) = TA \sum_i \lambda(v_i)v_i + \mathbf{b} \]
\[ = \sum_i \lambda(v_i)TAv_i + \sum_i \lambda(v_i)b \]
\[ = \sum_i \lambda(v_i)(TAv_i + b) \]
\[ = \sum_i \lambda(T(v_i))T(v_i) \]
\[ = \mathbf{x}(T(\Delta)), \]
and hence \(\chi\) is equivariant under similarity transformations. In the last step, Lemma 33(b) is used.

Then \((\lambda(v_0), \lambda(v_1), \lambda(v_2), \lambda(v_3))\) form the barycentric coordinates of the tetrahedron centre \(\chi\), and \(\lambda\) will be called the barycentric function of the tetrahedron centre \(\chi\).

In fact, barycentric functions generate all tetrahedron centres.
**Theorem 39.** (Barycentric functions generate all tetrahedron centres) If $\chi/\mathbf{x}$ is a tetrahedron centre of the form

$$\mathbf{x}(\Delta) := \sum_{i} \lambda_i(\Delta)\mathbf{v}_i \quad \text{for } \Delta = [V_0, V_1, V_2, V_3] \in S^3,$$

where $\lambda_0, \lambda_1, \lambda_2, \lambda_3 : S^3 \to \mathbb{R}$ are functions such that

$$\lambda_0 + \lambda_1 + \lambda_2 + \lambda_3 = 1,$$

then,

$$\lambda_i(\Delta) = \lambda(\mathbf{v}_i)$$

where $\lambda : \mathbb{V} \to \mathbb{R}$ is a barycentric function.

**Proof.** Writing

$$\lambda_0(\Delta) = \lambda_0(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3), \lambda_1(\Delta) = \lambda_1(\mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3),$$

$$\lambda_2(\Delta) = \lambda_2(\mathbf{v}_2, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3), \lambda_3(\Delta) = \lambda_3(\mathbf{v}_3, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)$$

and $\mathbf{x}(\Delta) = \mathbf{x}(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ as $\Delta$ is determined by its vertices $V_0, V_1, V_2, V_3$, we have

$$\mathbf{x}(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \sum_{i} \lambda_i(\Delta)\mathbf{v}_i$$

$$= \lambda_0(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_0 + \lambda_1(\mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_1 + \lambda_2(\mathbf{v}_2, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_3)\mathbf{v}_2 + \lambda_3(\mathbf{v}_3, \mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2)\mathbf{v}_3$$

for $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in V$ (76)

Since $\Delta$ is independent of the order of $\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, we can swap $\mathbf{v}_0$ and $\mathbf{v}_1$ to obtain

$$\mathbf{x}(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \mathbf{x}(\mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3)$$

$$= \lambda_0(\mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_1 + \lambda_1(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)\mathbf{v}_0 + \lambda_2(\mathbf{v}_2, \mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_3)\mathbf{v}_2 + \lambda_3(\mathbf{v}_3, \mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2)\mathbf{v}_3$$ (77)

Comparing the coefficients of $\mathbf{v}_0$ and $\mathbf{v}_1$ with those in (76), we have

$$\lambda_0(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \lambda_1(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \quad \text{and} \quad \lambda_1(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) = \lambda_0(\mathbf{v}_1, \mathbf{v}_0, \mathbf{v}_2, \mathbf{v}_3),$$

both of which suggest that $\lambda_0 = \lambda_1$, as the above relations hold for all $(\mathbf{v}_0, \mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3) \in V$. Similarly, by swapping $\mathbf{v}_1$ and $\mathbf{v}_2$ and by swapping $\mathbf{v}_2$ and $\mathbf{v}_3$, we will see that $\lambda_1 = \lambda_2$ and $\lambda_2 = \lambda_3$ respectively. Hence,

$$\lambda_0 = \lambda_1 = \lambda_2 = \lambda_3 = \lambda$$

for some function $\lambda : \mathbb{V} \to \mathbb{R}$, and we have to verify that $\lambda$ is a barycentric function.
In terms of $\lambda$, (76) and (77) can be written as
\[
\begin{align*}
x(v_0, v_1, v_2, v_3) &= \lambda(v_0, v_1, v_2, v_3)v_0 + \lambda(v_1, v_0, v_2, v_3)v_1 \\
&\quad + \lambda(v_2, v_0, v_1, v_3)v_2 + \lambda(v_3, v_0, v_1, v_2)v_3 \quad \text{and (78)} \\
x(v_1, v_0, v_2, v_3) &= \lambda(v_1, v_0, v_2, v_3)v_1 + \lambda(v_0, v_1, v_2, v_3)v_0 \\
&\quad + \lambda(v_2, v_1, v_0, v_3)v_2 + \lambda(v_3, v_1, v_0, v_2)v_3 \quad \text{and (79)}
\end{align*}
\]
respectively. Comparing the coefficients of $v_2$ and $v_3$ in (78) and (79), we see that $\lambda$ is symmetric in the second and third variables. Hence $\lambda$ is symmetric in the second, third and fourth variables, and the symmetry condition (ii) for a barycentric function is satisfied.

With the symmetry just proved, write $\lambda(v_i, v_j, v_k, v_l)$ as $\lambda(v_i)$ for $\{i, j, k, l\} = \{0, 1, 2, 3\}$. Then, the given condition (75) can be written as
\[
\lambda(v_0) + \lambda(v_1) + \lambda(v_2) + \lambda(v_3) = 1 \quad \text{(80)}
\]
matching the normalization condition (iii) for a barycentric function.

Let $T$ be any similarity transformation $T$ as described in (64). Since (80) holds for all affinely independent $\{v_0, v_1, v_2, v_3\}$, we will also have
\[
\lambda(T(v_0)) + \lambda(T(v_1)) + \lambda(T(v_2)) + \lambda(T(v_3)) = 1
\]
as $\{T(v_0), T(v_1), T(v_2), T(v_3)\}$ are also affinely independent according to Lemma 33(a). Then,
\[
T(x(\Delta)) = x(T(\Delta))
\]
\[
tA \sum_i \lambda(v_i)v_i + b = \sum_i \lambda(T(v_i))(tAv_i + b)
\]
\[
= tA \sum_i \lambda(T(v_i))v_i + \sum_i \lambda(T(v_i))b
\]
\[
tA \sum_i \lambda(v_i)v_i = tA \sum_i \lambda(T(v_i))v_i
\]
\[
\sum_i \lambda(v_i)v_i = \sum_i \lambda(T(v_i))v_i
\]
as $t \neq 0$ and $A$ is invertible. By comparing the coefficients of $v_0$ again, we have
\[
\lambda(v_0) = \lambda(T(v_0))
\]
and the invariance condition (i) for a barycentric function is satisfied. \qed
6. Summary

In this paper, we have accomplished the aims and objectives stated in Section 1.2. In Section 2, we have presented new characterizations of the classical triangle centres, namely centroid, circumcentre, incentre, excentre and orthocentre, and have proved their properties that carry over to tetrahedra.

In Section 3, we have generalized Monge point of tetrahedra to a family of tetrahedron centres lying on the Euler lines. As a consequence, Monge point and twelve-point centre of tetrahedra have been shown to share the common geometric feature of being the points of concurrence of special lines derived from their triangle counterparts.

In Section 4, we have constructed new generalizations of orthocentre of triangles to tetrahedra, namely antimedial circumcentre and orthic inexcentre. We did not merely define some new points, but have actually, and most importantly, found the geometric and algebraic properties that carry over to tetrahedra through these generalizations. More precisely, the homothethy between a triangle and its antimedial triangle, as well as that between its tangential and orthic triangles are preserved. The collinear of orthocentre and circumcentre with centroid or $\chi_{25}$, as well as the vector representations of orthocentre are preserved.

In Section 5, we have built a framework to study tetrahedron centres in general. While our definition of tetrahedron centre is geometric in nature, we have also found its algebraic representation in terms of barycentric function.

During this research, we have also observed signs of feasibility to extend all our work to higher-dimensional simplices. It is because our analytic approach requires only basic linear algebra, and we have intentionally avoided the use of cross product throughout. We can even expect synthetic proofs in higher-dimensions similar to those presented in Lemma 25. But due to our limited knowledge of higher-dimensional simplices, hyperplanes and hyperspheres, we were unready for such ambition at the current stage. We hope that after acquiring the necessary knowledge, we will revisit this problem and explore more tetrahedron centres in the future.

REFERENCES


Reviewer’s Comments

The paper under review studies generalizations of various classical centres of triangles (centroid, circumcentre, incentre, excentre and orthocentre) to those of tetrahedra. The authors start with the generalizations of first four centres, which are straightforward, by describing them as points of concurrence of certain straight lines (note that the corresponding centres of a triangle are also defined to be points of concurrence of some straight lines associated to the triangle). When it comes to orthocentres, no generalizations as straightforward as those of the other centres are available, as the three heights of a generic tetrahedron are not concurrent. It is for this reason that the authors devote a major part of the paper to studying a number of notions of orthocentres of a tetrahedron.

1. The Monge point and quasi-orthocentres, which form a family of points lying on the Euler line, the line joining the centroid and circumcentre of a tetrahedron, parametrized by the ratio of division. This generalizes the classical result that the orthocentre of a triangle lies on the line which joins the circumcentre and the centroid of the triangle.

2. Antimedial circumcentre, which is defined to be the circumcentre of the tetrahedron whose four sides are tangent to the circumsphere of the original tetrahedron. This definition is inspired by the result that the orthocentre of a triangle is the circumcentre of the triangle whose incircle is the circumcircle of the original triangle. This centre is represented as the sum of the position vectors of the four vertices if the origin is taken to be the circumcentre (Theorem 22).

3. Orthic incentre, defined as the incentre or excentre of a certain tetrahedron which is homothetic to the tetrahedron whose in-sphere is the circumsphere of the original tetrahedron. The orthic incentre is represented as the sum of the position vectors of the circumcentres of the four triangular sides of the tetrahedron (Theorem 29), and lies on the line joining the circumcentre and the centre of homothety (Theorem 28).

Finally, the authors define general tetrahedron centres intrinsically as a map from the set of tetrahedron to $\mathbb{R}^3$ which is equivariant with respect to rigid motions. Then they characterize general tetrahedron centres as linear combinations of the position vectors of vertices, where the coefficients are barycentric functions (Theorem 39). Thus the various tetrahedron centres studied before are special examples of general tetrahedron centres under this characterization.

The paper is well-written and organized, with ample illustrations and motivations for generalizations to tetrahedron centres well-explained. However, it appears that the mathematics involved in this paper is elementary and well-known (vectors and some high-school geometry). I would like to see more results of centres of higher dimensional simplices (e.g. barycentric description of centres) rather than multiple
proofs of the same elementary result (e.g. Lemma 25). The following are some specific comments on the papers.

1. p.4, last two lines: the sentence should read ‘...we will use a unified approach to generalize the classical triangle centres to those of tetrahedra by generalizing the vertices of a triangle to those of a tetrahedron and generalizing the edges...to the faces of a tetrahedron’.
2. p.26, the first paragraph: it should be placed after Definition 23 as otherwise tangent planes, which are mentioned there, have not been defined.
3. p.27, line -3, first sentence: it should read ‘Beware of the terminologies...by angle sizes...’.
4. p.28, Lemma 25: it is unnecessary to give three different proofs to a lemma, which is not one of the main results in the paper.
5. p.38, Definition 32: the notation $S^2$ is usually reserved for spheres. It is better to use another notation for the set of all triangles.
6. p.38, line 12: ‘lies’ should read ‘lie’.
7. p.38, line 16: ‘equivariances’ should read ‘equivariance’. Then ‘require’ should read ‘requires’.
8. p.38, line 19: ‘...are also a triangle...’ should read ‘...be also a triangle...’.
10. p.39, line of proof of Proposition 34: add $X_{25}$ after $K$.
11. p.43, Definition 37: ‘quadraples’ should read ‘quadruples’.
12. p.45: it would be better if they could spell out what the barycentric functions are for special examples of tetrahedron centres after the proof of Theorem 39.